

COMBINATORIAL CURVATURES, GROUP ACTIONS, AND COLOURINGS: ASPECTS OF TOPOLOGICAL COMBINATORICS

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Von der Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation

Promotionsausschuss:

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Tag der wissenschaftlichen Aussprache: 25. November 2004

Berlin 2005
D 83

CONTENTS

I	Curvature and Combinatorics	1
1	Foundations	9
1.1	Notation and Basic Definitions	10
1.2	An example: The 3-dimensional Cube	15
1.3	The Difference Operator	22
1.4	The Bochner-Laplacian	25
1.5	Combinatorial Curvature Tensors	27
2	Applications	33
2.1	Combinatorial Weitzenböck Formulae	34
2.2	Combinatorial Gauß-Bonnet-Formula for Surfaces	35
2.3	A combinatorial version of Bochner's Theorem for 1-chains	37
2.4	A second example and Bochner's Theorem for 1-chains	39
2.5	Problems with p -chains.	41
2.6	Unique Continuation Theorems for 2-chains.	42
2.7	Diameter estimates for some simple manifolds.	50
II	Topology and Combinatorics	61
3	Chromatic numbers of graphs	67
3.1	Preliminaries	69
3.2	Shore subdivision and useful subcomplexes	71
3.3	$L(G)$ as a \mathbb{Z}_2 -deformation retract of $B(G)$	73
3.4	The $K_{l,m}$ -theorem	75
4	Generalised Kneser colourings	77
4.1	Preliminaries	79
4.2	Examples and counterexamples	81
4.3	Groups acting on simplicial complexes	84
4.4	A topological lower bound for the chromatic number of hypergraphs	86
4.5	A combinatorial lower bound for Kneser hypergraphs with multiplicities . .	87
	Bibliography	91

Part I

Curvature and Combinatorics

INTRODUCTION

In the first part of this thesis we discuss a new notion of curvature for cell complexes, introduced by Forman [26], and some variations of it. This notion is purely combinatorial and uses local data to define a *combinatorial curvature*, enabling us to deduce global topological properties. Before we outline Part I, we point out important differences to earlier attempts to extend the notion of curvature to cell complexes. Common to these approaches is the fundamental assumption that the complex is either embedded in some euclidean space or itself a metric space that is isometric to a manifold. This enables one to measure angles, lengths, or volumes or to consider angle defects, dihedral angles, etc. This direction has been pursued by Cheeger, Müller, and Schrader [18] generalising Regge's earlier work [55], by Banchoff [7], by Alexandrov and Zalgaller [1], by Bobenko and Pinkall [14], and by many others. Recently, Hirani [32] described an exterior calculus for simplicial manifolds. Another approach is via characteristic classes: Combinatorial formulae for characteristic classes can be interpreted by analogy from the Riemannian setting as combinatorial formulae for curvatures; see [27, 30, 31].

Forman's approach differs significantly from the ones listed above. He considers abstract cell complexes. Weights can be associated to the cells of such a complex: They are introduced via an inner product that is defined on the \mathbb{R} -vector space spanned by the p -cells by the requirement that two distinct cells are orthogonal and the inner product of a p -cell β with itself is a non-zero number $w_{p,\beta}$, the *weight of β* . These weights assigned to cells are not interpreted as lengths, volumes, or angles. This might be disappointing for a geometer, but Forman's main goal is to analyse the combinatorial Hodge-Laplace matrix for cell complexes and he aims to derive global topological information from this analysis which is inspired by methods from global analysis. Therefore, the *Leitmotiv* is to ask to what extent ideas from global analysis can be used in an abstract setting of discrete geometry. The most prominent method to this respect in global analysis is *Bochner's method*. Bochner published a sequel of three articles between 1946 and 1949 [15, 16, 17], in which a formula of Weitzenböck [66] from 1923 is rediscovered and used for the first time to obtain global information. Generalisations of this formula are very important in differential geometry, e.g. for spin manifolds and Dirac operators, see Lawson and Michelsohn [42].

In the combinatorial setting, the p^{th} Hodge-Laplace matrix $\Delta_p = \delta\partial + \partial\delta$ is defined via the boundary and coboundary maps and assigns p -chains to p -chains. In [26], Forman postulated a combinatorial analogue of Weitzenböck's classical formula by a canonical decomposition of the combinatorial Hodge-Laplacian. The best way to explain this decomposition is by example. If the combinatorial Hodge-Laplacian Δ_p is a (3×3) -matrix, then we consider

$$\Delta_p = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = \begin{pmatrix} |b|+|c| & b & c \\ b & |b|+|e| & e \\ c & e & |c|+|e| \end{pmatrix} + \begin{pmatrix} a-(|b|+|c|) & 0 & 0 \\ 0 & d-(|b|+|e|) & 0 \\ 0 & 0 & f-(|c|+|e|) \end{pmatrix} =: \Delta^F + \text{Ric}^F.$$

The diagonal entry $(\text{Ric}^F)_{ii}$ defines the *combinatorial Ricci curvature* of the p -cell i . For $p = 1$, Forman calls this formula a *combinatorial Weitzenböck formula* since we have the following analogy to Weitzenböck's formulae on a Riemannian manifold M . The difference

between the Hodge-Laplacian (or the Laplace-Beltrami) Δ_p and the *rough Laplacian* (as it is called by Berger [12]) $\Delta_p^\nabla := \nabla^* \nabla$ on p -forms can be expressed in terms of the Riemannian curvature tensor \mathcal{R} :

$$\Delta_p = \Delta_p^\nabla + F_p(\mathcal{R}).$$

We note that there are different names for the rough Laplacian in the literature. For example, it is called *connection Laplacian* by Lawson and Michelsohn [42]. The classical result of Weitzenböck and others is that $F_0(\mathcal{R}) = 0$ and $F_1(\mathcal{R}) = \text{Ric}$, the Ricci curvature of M considered as a $(1, 1)$ -tensor. Explicit expressions for larger p become more involved, compare Li [43] or Jost [33]. Forman's justification to call this decomposition *combinatorial Weitzenböck formula* and the curvature term involved *combinatorial Ricci curvature* is rather weak. As Forman puts it [26],

“Our next step is to develop a Weitzenböck formula. [...] In the combinatorial setting this is rather mysterious, since we begin by knowing neither a combinatorial analogue for $\nabla^* \nabla$ nor for F_p .”

So the justification for the names stems less from a computation of $\nabla^* \nabla$ or a Ricci curvature tensor than from the following theorems. These theorems are combinatorial analogues of Bochner's theorem and Myers' theorem for Riemannian manifolds.

- *Combinatorial theorem of Myers, [26, Theorem 6.1].*

Let X be a finite *quasiconvex* complex and assign a *standard set of weights* to its cells. Suppose that we have $\text{Ric}^F(e) > 0$ for every edge e . Then $\pi_1(X)$ is finite.

- *Combinatorial theorem of Bochner for 1-chains, [26, Section 4].*

Suppose M is a finite, connected, combinatorial n -manifold such that $\text{Ric}^F(e) \geq 0$ for all edges e . If $n \leq 3$ then the first Betti number of M is at most n . If $n > 3$ and the dual complex contains a combinatorial n -simplex or n -cube then the first Betti number of M is at most n .

Myers' theorem in differential geometry says a bit more. If the Ricci curvature of a Riemannian manifold (M, g) has a positive lower bound, then a sphere of a certain radius that depends on this curvature bound yields an upper bound for the diameter of (M, g) . A diameter of a (not necessarily finite) CW-complex X can be defined as follows. An *edge-path* between two vertices is a sequence of 1-cells such that two consecutive edges have a common vertex, its *length* is the number of edges used. The *distance* between two vertices is the length of a shortest edge-path between these vertices if it is finite and infinite otherwise. The *diameter* of X is now the supremum of the distance between any two vertices of X . Forman [26, Theorem 6.3] proves an upper bound for the diameter of a quasiconvex CW-complex X if there is a positive c such that $\text{Ric}^F(e) \geq c$ for all edges e of X . But so far there is no interpretation of this inequality as a comparison with a model space that depends on the curvature. In general, it is very difficult to obtain diameter bounds for combinatorial manifolds. Even in the special case of convex polytopes, that is, certain combinatorial manifolds that are spheres, not much is known so far. But diameters of convex polytopes are of great interest in linear programming, since complexity issues of the

simplex algorithm are connected to diameter bounds of convex polytopes. A long-standing conjecture in this respect is the *Hirsch conjecture*, which asks for an upper bound on the diameter of a d -polytope with n facets. We come back to this question at the end of Chapter 2, where we study diameter estimates for combinatorial manifolds. Finally, Forman proves the following statement:

- *Existence of everywhere negatively Ricci curved subdivisions, Section 7 of [26].*

Let M be a simplicial combinatorial manifold of dimension at least two. Then there is a subdivision M' of M such that for every edge e of M' we have $\text{Ric}(e) < 0$ for a *standard set of weights*.

This can be seen as a combinatorial analogue of theorems due to Gao [28], Gao and Yau [29], and Lohkamp [44, 45]: Every smooth manifold M of dimension at least three admits a Riemannian metric with everywhere negative Ricci curvature. A crucial difference between the combinatorial and the smooth result is that the combinatorial version is true in dimension two while the smooth version is false in dimension two: The smooth sphere and torus do not admit a Riemannian metric that has an everywhere negative Ricci curvature. This fact follows from the well-known theorem of Gauß and Bonnet. Forman's result can be rephrased as follows: It is impossible to prove a combinatorial analogue of the Gauß–Bonnet theorem,

$$\sum_{e \text{ edge of } M} \text{Ric}^F(e) = \lambda \cdot \chi(M)$$

for a non-negative constant λ and a *standard set of weights*. In contrast, we shall show in Chapter 2 that the situation changes dramatically if we consider a modified notion Ric of combinatorial Ricci curvature introduced in Chapter 1 *and* choose certain weights, *the geometric set of weights*. In this setting, we are able to prove a combinatorial analogue of the Gauß–Bonnet theorem:

$$\sum_{e \text{ edge of } M} \text{Ric}(e) = 4 \cdot \chi(M).$$

We remark that Ric and Ric^F coincide if a standard set of weights is chosen.

Another aim that we pursue in this thesis is to derive the Bochner-Laplacian Δ^F and the combinatorial Ricci curvature Ric^F by methods inspired from differential geometry in order to derive the combinatorial version of Weitzenböck's formula. To this direction, we use some naïve discrete bundle theory. This goes beyond the exterior calculus described for example by Hirani [32]. The *classical approach* in a discretised manifold theory is to consider p -chains and cochains as the appropriate objects but one does not consider any bundles. No doubt, this classical approach is extremely valuable: As we know from de Rham's classical theory, there are for example deep and important connections between (harmonic) p -cochains of a cell decomposition of a smooth manifold and (harmonic) p -forms. But to what extent can the useful concept of fibre bundles be used in the discrete setting? A naïve approach in this direction is the following: Fix a vector space V and consider a copy of this space as fibre for each p -cell. A function on a CW-complex M is a p -cochain (every cell is assigned the corresponding coefficient of the cochain) and a section in

a *discrete vector bundle* over M with fibre V is defined on the p -cells as a map that assigns a vector $v \in V$ to each p -cell. Natural choices as fibre at a p -cell are $C_p(M; \mathbb{R})$, $C^p(M; \mathbb{R})$, or tensor products of these spaces. Roughly speaking, this is the initial approach we take. Nevertheless many technical problems have to be solved and methods from differential geometry do not translate literally to the discrete setting.

Inspired from the smooth setting and its covariant derivative, we define a *combinatorial difference operator* ∇ and derive a *condensed combinatorial Bochner-Laplacian* Δ^∇ and a *condensed combinatorial Ricci curvature* Ric for weighted complexes from it. The constructions we use, once we have an analogue for the the covariant derivative, are again influenced by ideas from differential geometry. But one should be aware that we do not carry all properties from differential geometry to combinatorial geometry. For example there is no *discrete Leibniz rule* for ∇ and the combinatorial Riemannian curvature tensor does not have the (skew)symmetries one often uses in the smooth world. In the combinatorial setting, we take differences at a cell in direction of a neighbouring cell. Since the combinatorics changes from cell to cell in general, we do not have a proper combinatorial analogue of parallel transport.

The first problem we face is that in differential geometry a vector field can be differentiated in two distinct ways: Via flows (and therefore parallel transport) and via a *Levi-Civita connection*. Both notions are closely related to each other. We define a combinatorial analogue of the Levi-Civita connection. In contrast to differential geometry, this combinatorial difference operator is defined fibrewise, that is, if we have two sections in a discrete fibre bundle with fibre $C_p(M; \mathbb{F})$, we compute the differences $\nabla_Y X$ fibrewise. Let us consider an example. At a given p -cell β we consider $(\nabla_{Y|_\beta} X|_\beta)|_\beta$. In particular, this value is determined by the values of X and Y at β . It does not depend on the other values of X in a neighbourhood of β as we might be tempted to expect from our experience in the smooth setting. Moreover, we want that *second order differences* commute, that is, $\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z$. This is achieved by definition in Section 1.3.

The second problem arises if we have a closer look to the left hand side and right hand side of the combinatorial version of Weitzenböck's formula mentioned above. The combinatorial Hodge-Laplacian operates on functions, that is, on p -chains or p -cochains (we identify chains and cochains by dualisation). In the smooth setting, the rough Laplacian operates on 1-forms. So in the combinatorial setting, we would like to consider sections that assign a p -(co)chain to each p -cell. Since a p -(co)chain α can be seen as a constant section where α is assigned to each p -cell, we obtain from a p -chain a constant section and make our computations fibrewise for this section. This means essentially to apply in each fibre over β a linear map $A|_\beta$. The resulting section can then be interpreted again as p -(co)chain.

We now outline the content and organisation of Part I of this thesis which is divided into two chapters: *Foundations* and *Applications*.

Chapter 1 starts in Section 1.1 with an introduction to the notation and provides some fundamental facts needed in the following about weighted CW-complexes, their Hodge-Laplacian Δ , and tensors on CW-complexes. In Section 1.2 we give an overview on the technical details of Chapter 1 by a simple example: We compute the difference operator,

the Bochner-Laplacian, and the Ricci curvature tensor for this example although the definitions for these objects follow in Sections 1.3–1.5. The fundamental concept of a *difference mapping* and a *difference operator* ∇ is defined in Section 1.3. This tool is used in Section 1.4 to construct and explicitly describe a combinatorial analogue of a rough Laplacian Δ^∇ that we call Bochner-Laplacian in the discrete setting. In Section 1.5 we define combinatorial analogues of the Riemannian curvature tensor and its trace, the Ricci tensor Ric . Some rather technical but straight-forward computations yield a detailed description of the Ricci tensor.

Chapter 2 is devoted to applications of the objects described in Chapter 1. Weitzenböck's formula $\Delta = \Delta^\nabla + \text{Ric}$ is proved in Section 2.1 for any choice of non-zero weights. This formula differs from Forman's formula in general, but if a *standard set of weights* is assigned to the cells of a cell complex, both decompositions of the Hodge-Laplacian coincide. In Section 2.2 we use the *geometric set of weights* to obtain a combinatorial version of the classical Gauß–Bonnet theorem for closed surfaces. As indicated earlier, the geometric set of weights yield examples with $\Delta^F \neq \Delta^\nabla$ and $\text{Ric}^F \neq \text{Ric}$. In Section 2.3 and Section 2.4, we summarise Forman's proof of a combinatorial version of Bochner's theorem for 1-chains and consider two different cell decompositions of the 2-dimensional torus with a standard set of weights and the geometric set of weights. In [26, Section 5], Forman describes difficulties that appear if one tries to extend the combinatorial version of Bochner's theorem from 1-chains to p -chains; we briefly present and discuss them in Section 2.5. To prove the combinatorial version of Bochner's theorem for 1-chains, two theorems are crucial: A *unique continuation theorem* that tells us that a locally vanishing 1-chain contained in $\text{Ker } \delta \cap \text{Ker } \Delta^\nabla$ vanishes globally and a theorem that exhibits the *homology dimension* (as defined in Section 2.3) of a non-negatively Ricci-curved closed combinatorial manifold M as an upper bound of the first Betti number of M . In Section 2.6 we describe possible generalisations of a *unique continuation theorem for 2-chains* and problems that arise with these. On combinatorial manifolds of dimension larger than three, an additional assumption must be made that leads to problems if we try to prove that the *homological dimension* is an upper bound for the second Betti number. The chapter ends in Section 2.7 with a discussion on diameter estimates of a *simple closed combinatorial d -manifold*, that is, a closed combinatorial manifold which is dual to a closed simplicial combinatorial manifold. We follow Forman's proof of a combinatorial version of Myers' theorem for arbitrary combinatorial manifolds and extend his approach from a standard set of weights to more general weights. We restrict to simple manifolds to simplify the presentation. Technical complications occur in case of non-simple manifolds, but these can be solved as described by Forman [26]. The aim is to obtain a diameter estimate not only for positively Ricci-curved manifolds with respect to a *standard set of weights* (as done by Forman) but for positively Ricci-curved manifolds with respect to other non-zero weights. We give some simple examples to show that this is in fact possible.

CHAPTER 1

FOUNDATIONS

INTRODUCTION

The programme of this chapter is as follows: We introduce a *combinatorial difference operator* ∇ in Section 1.3 that serves as a combinatorial analogue of a covariant derivative in differential geometry on weighted quasiconvex CW-complexes. This difference operator is then employed to tailor other combinatorial objects in accordance with patterns from differential geometry:

- An *extended combinatorial Bochner-Laplacian* $\tilde{\Delta}^\nabla = \nabla^* \nabla$ in Section 1.4.
- An *extended combinatorial version of the Riemannian curvature tensor* \mathcal{R} in Section 1.5.
- An *extended combinatorial Ricci curvature tensor* $\widehat{\text{Ric}}$ that is a trace of \mathcal{R} in Section 1.5.

All computations involved are rather technical. In order to familiarise the reader with the definition of the difference operator ∇ and to illustrate the computations of Section 1.4 and 1.5, we give an example in Section 1.2: We briefly explain the difference operator of the boundary of a 3-dimensional cube with a *geometric set of weights* chosen and show how to obtain the Bochner-Laplacian *the hard way* by direct computation and the *easy way* by using Lemma 1.4.2. We finish this example by reading off the combinatorial Ricci curvature by Corollary 1.5.8.

But things are a bit more complicated: The extended Bochner-Laplacian and extended Ricci curvature carry too much information. They are *tensors*, that is, they map objects that assign each p -cell a p -chain to objects that assign each p -cell a p -chain. Instead, we want to relate such a tensor to an operator that maps p -chains to p -chains. Such a transformation can in fact be done. The information that is relevant consists of the diagonals of these extended mappings at each cell. Denote the number of p -cells by f_p . The f_p diagonals of the f_p extended Bochner-Laplacians (resp. extended Ricci curvatures) form the f_p rows of the *condensed Bochner-Laplacian* Δ^∇ (resp. *condensed Ricci curvature* $\widehat{\text{Ric}}$). These *condensed* objects are the objects we are aiming at. Moreover, they coincide with the Bochner-Laplacian Δ^F and Ricci curvature Ric^F of Forman for important classes of weights chosen for the CW-complex, e.g., if a *standard set of weights* is assigned to the cells.

1.1 NOTATION AND BASIC DEFINITIONS

In this section we recall the definitions of *quasiconvex CW-complexes*, of the *neighbourhood relation* of p -cells, and of *weighted CW-complexes* and their *Hodge-Laplacian*. Moreover, we introduce special classes of weights for weighted complexes. Finally, we define a concept of *tensors on CW-complexes* that resembles many properties of tensor fields on smooth manifolds.

Quasiconvex complexes: For a finite and regular CW-complex M we denote the set of p -cells by K_p and its cardinality by f_p . We refer to Munkres [52], Cooke and Finney [20], or Lundell and Weingram [47] for an introduction to (finite and regular) CW-complexes. The set $\{1, \dots, f_p\}$ is denoted by $[f_p]$. We restrict ourselves to *quasiconvex complexes*: For each pair of distinct p -cells β_1 and β_2 such that the intersection I of their closures contains a $(p-1)$ -cell α , I is the closure of α .

Neighbourhood relation: Two p -cells $\beta_1 \neq \beta_2$ are called *neighbours* if there is a $(p+1)$ -cell γ such that β_1 and β_2 are faces of γ (shorthand notation: $\beta_1, \beta_2 < \gamma$) or there is a $(p-1)$ -cell α that is a face of β_1 and β_2 . The neighbours β_1 and β_2 are *transverse neighbours*, $\beta_1 \pitchfork \beta_2$, if there are a $(p+1)$ -cell γ and a $(p-1)$ -cell α satisfying the conditions above. We also write $\beta_1 \pitchfork_\alpha^\gamma \beta_2$. They are *parallel neighbours*, $\beta_1 \parallel \beta_2$, if they are not transverse. To indicate the connecting cell, we write $\beta_1 \parallel^\gamma \beta_2$ or $\beta_1 \parallel_\alpha \beta_2$. Examples for the different types of neighbours occur in the CW-complex of Figure 1.1: The cells β_0 and β_4 are not neighbours, since they do not share a common vertex and are not simultaneously contained in the boundary of a 2-cell. The transverse neighbours of β_4 are $\beta_2, \beta_3, \beta_5$, and β_7 , because we have $\beta_2 \pitchfork_{\alpha_2}^{\gamma_2} \beta_4$, $\beta_3 \pitchfork_{\alpha_4}^{\gamma_2} \beta_4$, $\beta_5 \pitchfork_{\alpha_5}^{\gamma_1} \beta_4$, and $\beta_7 \pitchfork_{\alpha_6}^{\gamma_1} \beta_4$. The parallel neighbours of β_4 are β_1 and β_6 , since $\beta_1 \parallel_{\alpha_1} \beta_4$ and $\beta_6 \parallel_{\alpha_6} \beta_4$.

We agree on the following convention: If we consider p -cells and their neighbours (which are also p -cells), we denote them by β_1, β_2, \dots . The $(p-1)$ -cells will be denoted by $\alpha_1, \alpha_2, \dots$ while the $(p+1)$ -cells will be denoted by $\gamma_1, \gamma_2, \dots$.

Weighted complexes: From now on, we fix an orientation for every cell of the CW-

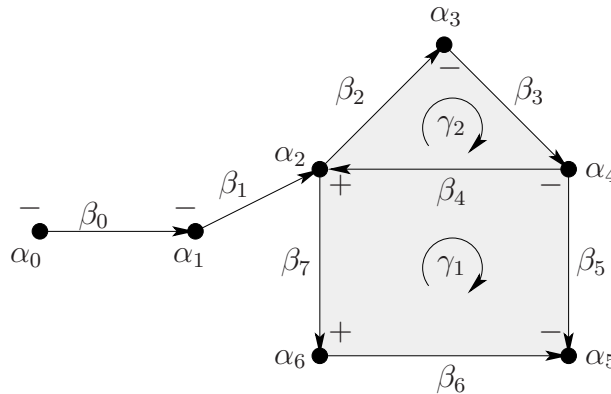


Figure 1.1: We illustrate the concept of parallel and transverse neighbours. Parallel neighbours of β_4 are β_1 and β_6 . Transverse neighbours of β_4 are $\beta_2, \beta_3, \beta_5$, and β_7 . The cell β_0 is not a neighbour of β_4 .

complex under consideration. A p -chain with coefficients in a field $\mathbb{F} \in \{\mathbb{R}; \mathbb{C}\}$ is an element of the \mathbb{F} -vector space $C_p(M; \mathbb{F})$ that has the (oriented) p -cells K_p as basis. On $C_p(M; \mathbb{F})$ we have a weighted standard inner product g defined by requiring all p -cells to be pairwise orthogonal and by \mathbb{F} -bilinear extension. If we use Kronecker's symbol δ_{jk} and if we indicate weighted cells by a tilde and their weights by $w_{p,j}$, we have

$$g(\tilde{\beta}_j, \tilde{\beta}_k) := w_{p,j} w_{p,k} \delta_{jk}.$$

There is a canonical boundary operator $\partial = \{\partial_p : C_p(M; \mathbb{F}) \longrightarrow C_{p-1}(M; \mathbb{F})\}$, which yields a chain complex $\{C_p(M; \mathbb{F}), \partial_p\}$. The matrix representation of the boundary map $\partial_p : C_p(M; \mathbb{F}) \longrightarrow C_{p-1}(M; \mathbb{F})$ can be read off from

$$\partial_p \tilde{\beta}_j = \sum_{k \in [f_{p-1}]} [\tilde{\beta}_j : \tilde{\alpha}_k] \tilde{\alpha}_k, \quad (1.1)$$

where $\tilde{\beta}_j \in K_p$ and the coefficient $[\tilde{\beta}_j : \tilde{\alpha}_k] \in \{\pm 1; 0\}$ equals the incidence number of the oriented cells $\tilde{\beta}_j$ and $\tilde{\alpha}_k$. The equation $[\tilde{\gamma} : \tilde{\beta}_r][\tilde{\beta}_r : \tilde{\alpha}] + [\tilde{\gamma} : \tilde{\beta}_s][\tilde{\beta}_s : \tilde{\alpha}] = 0$ encodes the boundary property $\partial_{p-1} \partial_p = 0$ if we assume all four incidence numbers to be non-zero. In this case the equation is equivalent to

$$[\tilde{\gamma} : \tilde{\beta}_r][\tilde{\gamma} : \tilde{\beta}_s] + [\tilde{\beta}_r : \tilde{\alpha}][\tilde{\beta}_s : \tilde{\alpha}] = 0. \quad (1.2)$$

Keep in mind that we are dealing with regular complexes only and compare Cooke and Finney [20] or Lundell and Weingram [47, Chapter 5] for details.

We are now interested in a *coboundary operator* δ . There is a canonical candidate, once we have chosen inner products for $C_p(M; \mathbb{F})$, $0 \leq p$. The natural way to view the coboundary operator δ_p is as a map between $C^{p-1}(M; \mathbb{F})$ (the dual of $C_{p-1}(M; \mathbb{F})$) and $C^p(M; \mathbb{F})$. Since we can identify $C^p(M; \mathbb{F})$ and $C_p(M; \mathbb{F})$ in a canonical way if we use the inner product g , we use a slightly different approach and consider the coboundary map $\delta_p : C_{p-1}(M; \mathbb{F}) \longrightarrow C_p(M; \mathbb{F})$ defined as

$$\delta_p \tilde{\alpha}_j := \sum_{k \in [f_p]} \left(\frac{w_{(p-1),j}}{w_{p,k}} \right)^2 [\tilde{\beta}_k : \tilde{\alpha}_j] \tilde{\beta}_k. \quad (1.3)$$

We now briefly explain why this is a suitable coboundary operator. We use the inner product g to translate between $C_p(M; \mathbb{F})$ and $C^p(M; \mathbb{F})$. A natural choice for a basis of $C^p(M; \mathbb{F})$ is as follows: Associate to any basis element $\tilde{\beta}_j$ of $C_p(M; \mathbb{F})$ the element of $C^p(M; \mathbb{F})$ that maps $\tilde{\beta}_j$ to 1, that is,

$$\tilde{\beta}_j \longmapsto \frac{1}{w_{p,j}^2} g(\tilde{\beta}_j, \cdot).$$

If set $\tilde{\beta}^j := \frac{1}{w_{p,j}^2} g(\tilde{\beta}_j, \cdot)$, then δ is described with respect to this basis of $C^p(M; \mathbb{F})$ by

$$\delta_{p+1} \tilde{\beta}^j = \sum_{k \in [f_{p+1}]} [\tilde{\gamma}_k : \tilde{\beta}_j] \tilde{\gamma}^k.$$

This is the commonly used coboundary operator for the cochain complex formed by the spaces $C^p(M; \mathbb{F})$. This identification of $C_p(M; \mathbb{F})$ and basis $\{\tilde{\beta}_1, \dots, \tilde{\beta}_{f_p}\}$ with $C^p(M; \mathbb{F})$ and basis $\{\tilde{\beta}^1, \dots, \tilde{\beta}^{f_p}\}$ is important. We use it to *raise* an index, that is, to translate a statement on chains into a statement on cochains, and to *lower* an index, that is, to translate a statement on cochains into a statement on chains. Raising and lowering indices will become important when we consider tensors. For this reason it is useful to use orthonormal bases for $C_p(M; \mathbb{F})$ and $C^p(M; \mathbb{F})$ ($C^p(M; \mathbb{F})$ has an inner product induced from $C_p(M; \mathbb{F})$). We shall consider $\beta_j := \frac{1}{w_{p,j}} \tilde{\beta}_j$ for $C_p(M; \mathbb{F})$ and $\beta^j := \frac{1}{w_{p,j}} g(\tilde{\beta}_j, \cdot)$ for $C^p(M; \mathbb{F})$. Unless otherwise stated, we shall always use these orthonormal bases for calculations and use the elements of these bases to indicate the cells. In particular, the boundary and coboundary operators have to be adopted in the obvious way.

Hodge-Laplacian: The p^{th} *combinatorial Hodge-Laplace operator* Δ_p is the endomorphism on $C_p(M; \mathbb{F})$ given by $\partial_{p+1}\delta_{p+1} + \delta_p\partial_p$. We give an explicit matrix representation of this map with respect to the (ordered) orthonormal basis $\beta_1, \dots, \beta_{f_p}$ of $C_p(M; \mathbb{F})$. The advantage of this basis is that Δ_p is represented by a symmetric matrix.

The matrix associated to the combinatorial Hodge-Laplacian Δ_p of a weighted quasi-convex CW-complex M is of the form

$$(\Delta_p)_{jk} = \begin{cases} \frac{w_{(p-1),\alpha}^2}{w_{p,j}w_{p,k}} [\beta_j : \alpha][\beta_k : \alpha] & \beta_j \parallel_{\alpha} \beta_k, \\ \frac{w_{p,j}w_{p,k}}{w_{(p+1),\gamma}^2} [\gamma : \beta_j][\gamma : \beta_k] & \beta_j \parallel^{\gamma} \beta_k, \\ \frac{w_{p,j}w_{p,k}}{w_{(p+1),\gamma}^2} [\gamma : \beta_j][\gamma : \beta_k] + \frac{w_{(p-1),\alpha}^2}{w_{p,j}w_{p,k}} [\beta_j : \alpha][\beta_k : \alpha] & \beta_j \cap_{\alpha}^{\gamma} \beta_k, \\ \sum_{\alpha < \beta_k} \left(\frac{w_{(p-1),\alpha}}{w_{p,k}} \right)^2 + \sum_{\gamma > \beta_k} \left(\frac{w_{p,k}}{w_{(p+1),\gamma}} \right)^2 & j = k, \\ 0 & \text{otherwise.} \end{cases} \quad (1.4)$$

One way to convince yourself that these formulae are correct is to apply Formulae 1.1 and 1.3 to compute the Hodge-Laplacian with respect to the basis $\{\tilde{\beta}_j\}$ and transform a change of basis to $\{\beta_j\}$. Another way is to describe the matrices of ∂ and δ with respect to the orthonormal bases and then compute the Hodge-Laplacian. An equivalent description is given by Forman [26, Equation 2.2]:

$$(\Delta_p)_{jk} = \sum_{\gamma \in K_{p+1}} \frac{w_{p,j}w_{p,k}}{w_{(p+1),\gamma}^2} [\gamma : \beta_j][\gamma : \beta_k] + \sum_{\alpha \in K_{p-1}} \frac{w_{(p-1),\alpha}^2}{w_{p,j}w_{p,k}} [\beta_j : \alpha][\beta_k : \alpha].$$

The intention of (1.4) is to show how the different types of neighbours determine the entries of (Δ_p) . This point of view will be useful in the subsequent sections.

Special classes of weights: Many of the formulae we shall describe hold for arbitrary positive weights. Nevertheless it helps sometimes to focus on special subclasses of weights. One reason is that some theorems only hold for certain classes. The combinatorial analogue of Myers' theorem and the existence of a subdivision such that every edge has negative

Ricci curvature Ric^F are such examples. Forman proved most theorems only in case of a *standard set of weights* defined below. Another reason is that the formulae may simplify significantly. This is true in the case of a standard set of weights which we now describe. Another important class is the *geometric set of weights*.

Definition 1.1.1 (standard set of weights).

A choice of weights $w_{p,\alpha}$ for all p -cells α and all p is called *standard set of weights* if we choose positive numbers κ_1 and κ_2 and weigh the cells according to their dimension: Let $w_p = \sqrt{\kappa_1 \cdot \kappa_2^p}$ be the weight associated to every p -cell.

For standard set of weights we have a simplified form of the Hodge-Laplacian, since $(\Delta_p)_{ij} = 0$ for transverse neighbours β_j and β_i , as one easily verifies by substitution into (1.4).

Definition 1.1.2 (geometric set of weights).

For surfaces (combinatorial 2-manifolds) or 2-dimensional (quasiconvex) cell complexes, we define the *geometric set of weights* by

$$w_{0,j} = \sqrt{\frac{1}{\deg(\alpha_j)}}, \quad w_{1,j} \equiv 1, \quad \text{and} \quad w_{2,j} = \sqrt{\text{sides}(\gamma_j)},$$

where the degree $\deg(\alpha)$ is the number of 1-cells incident to the vertex α and $\text{sides}(\gamma)$ the number of 1-cells contained in the boundary of the 2-cell γ .

The geometric set of weights is interesting not only because it can be used to prove a combinatorial analogue of the theorem of Gauß and Bonnet, but also because the resulting Hodge-Laplacian is closely related to random walks on the 1-skeleton, as demonstrated by Chung [19].

Tensors on CW-complexes: The concept of a tensor on a d -dimensional CW-complex M to be introduced now parallels some ideas from tensors of smooth manifolds: They are defined pointwise, the linear combination of two tensors of the the same type can be formed by pointwise linear combination, they transform according to a change of basis of $C_p(M; \mathbb{F})$ or $C^p(M; \mathbb{F})$ with $0 \leq p \leq d$, and the (tensor) product of two tensors is defined. For a tensor product, the ordering is important.

Let r, s be non-negative integers and $\mathbf{r}, \mathbf{s} \subseteq [r+s]$ a partition of $[r+s]$. A p^{th} tensor field of type (\mathbf{r}, \mathbf{s}) can be described in the language of discrete fibre bundles. The *base* is M and for each p -cell $\alpha \in M$ we have a *fibre* that is $\bigotimes_{j=1}^{r+s} V_j$, where $V_j = C_p(M; \mathbb{F})$ for $j \in \mathbf{r}$ and $V_j = C^p(M; \mathbb{F})$ for $j \in \mathbf{s}$. Obviously, other vector spaces could be used, but this will not be of relevance for us.

Definition 1.1.3. For integers $r \geq 0, s \geq 0$ not both zero we consider a partition of $[r+s]$ into a set \mathbf{r} of cardinality r and \mathbf{s} of cardinality s . Let α be a p -cell of the CW-complex M . A p^{th} tensor of type (\mathbf{r}, \mathbf{s}) at α is given by an \mathbb{F} -multilinear function

$$T|_{\alpha} = (\beta_1)|_{\alpha} \otimes \dots \otimes (\beta_{r+s})|_{\alpha} : \bigotimes_{j=1}^{r+s} V_j \longrightarrow \mathbb{F},$$

where $(\beta_j)_{|\alpha} \in (V_j)^*$ (the dual of V_j), $V_j = C_p(M; \mathbb{F})$ for $j \in \mathbf{r}$, and $V_j = C^p(M; \mathbb{F})$ for $j \in \mathbf{s}$.

A p^{th} tensor field T of type (\mathbf{r}, \mathbf{s}) is defined as follows:

$$T := \bigcup_{\alpha \in K_p} T_{|\alpha} = \bigcup_{\alpha \in K_p} (\beta_1)_{|\alpha} \otimes \dots \otimes (\beta_{r+s})_{|\alpha} : K_p(M) \times \bigotimes_{j=1}^{r+s} V_j \longrightarrow \mathbb{F},$$

where $T_{|\alpha}$ is of type (\mathbf{r}, \mathbf{s}) for each $\alpha \in K_p$.

We often refer to an (\mathbf{r}, \mathbf{s}) -tensor as an (r, s) -tensor if it is clear how the factors in the tensor product are ordered. Using the technique of raising and lowering indices, we can transform an (r, s) -tensor field into a tensor field of type $(r+1, s-1)$ or $(r-1, s+1)$. The necessary tools for this have been introduced when we defined the coboundary operator. For a tensor field of type (\mathbf{r}, \mathbf{s}) , *lowering an index* $k \in \mathbf{r}$ means that for every p -cell α the factor k of $\bigotimes_{j=1}^{r+s} V_j$ of $T_{|\alpha}$ is dualised. This procedure yields a tensor field of type $(\mathbf{r} \setminus \{k\}, \mathbf{s} \cup \{k\})$. Similarly, *raising an index* $k \in \mathbf{s}$ means that for every p -cell α the factor k of $\bigotimes_{j=1}^{r+s} V_j$ of $T_{|\alpha}$ is dualised to obtain a tensor field of type $(\mathbf{r} \cup \{k\}, \mathbf{s} \setminus \{k\})$. Recall that this dualisation includes a special choice of basis of the dual which is determined by the inner product on $C_p(M; \mathbb{F})$ and $C^p(M; \mathbb{F})$.

The inner products g of $C_p(M; \mathbb{F})$ and g' of $C^p(M; \mathbb{F})$ extend naturally to tensor products: The inner product \tilde{g} on $V_1 \otimes \dots \otimes V_r$ is defined as

$$\tilde{g}(\alpha_1 \otimes \dots \otimes \alpha_r, \beta_1 \otimes \dots \otimes \beta_r) := g_1(\alpha_1, \beta_1) \cdot \dots \cdot g_r(\alpha_r, \beta_r),$$

where $\alpha_j, \beta_j \in V_j$, V_j is either $C_p(M; \mathbb{F})$ or $C^p(M; \mathbb{F})$, and g_j is (depending on V_j) either g or g' .

Contraction of a tensor field: An important operation applied to tensors in differential geometry is contraction. We now discuss an analogue in our combinatorial setting. Let $u \in \mathbf{r}$ and $v \in \mathbf{s}$. Then for every p -cell α we have the chains and cochains $(\beta_u)_{|\alpha} \in C^p(M; \mathbb{F})$ and $(\beta_v)_{|\alpha} \in C_p(M; \mathbb{F})$ that correspond to the chain and cochain specified by u and v of $T_{|\alpha}$. Analogy from the smooth category tempts us to consider the following cellwise (u, v) -contraction $\tilde{C}^{u,v}T$ of T : Omit factors u and v to obtain the $(r-1, s-1)$ -tensor $\tilde{T}_{|\alpha}$ at α and rescale $\tilde{T}_{|\alpha}$ by $(\beta_u)_{|\alpha} ((\beta_v)_{|\alpha})$:

$$(\tilde{C}^{u,v}T)_{|\alpha} := (\beta_u)_{|\alpha} ((\beta_v)_{|\alpha}) \tilde{T}_{|\alpha}$$

This first intuition has to be refined a little bit. In the combinatorial setting we have to take the base point into account where we contract: Instead of the cochain $(\beta_u)_{|\alpha}$ we shall use a projection $\pi((\beta_u)_{|\alpha})$ of $(\beta_u)_{|\alpha}$ for which the coefficient of the cocell α^* dual to α is zero. Let \mathbf{r} and \mathbf{s} be a partition of $[r+s]$ for $r, s > 0$ and choose $u \in \mathbf{r}$ and $v \in \mathbf{s}$. To give a concise definition of contraction, we introduce the function $f_{u,v} : [r+s-2] \longrightarrow [r+s]$ that *expands* $[r+s-2]$ to $[r+s]$ by skipping u and v :

$$f_{u,v}(x) = \begin{cases} x & x < \min\{u, v\} \\ x+1 & \min\{u, v\} \leq x < \max\{u, v\} \\ x+2 & x \geq \max\{u, v\}. \end{cases}$$

Definition 1.1.4. Let $T = \beta_1 \otimes \dots \otimes \beta_{r+s}$ be a tensor field of type (\mathbf{r}, \mathbf{s}) with $r > 0$ and $s > 0$ on a CW-complex M . Choose $u \in \mathbf{r}$ and $v \in \mathbf{s}$. For a p -cell α define the projection of $(\beta_u)_{|\alpha}$ as $\pi((\beta_u)_{|\alpha}) := (\beta_u)_{|\alpha} - (\beta_u)_{|\alpha}(\alpha)\alpha^*$. We (u, v) -contract T to obtain the tensor field $C^{u,v}T$ of type $(r-1, s-1)$ as follows:

$$(C^{u,v}T)_{|\alpha} := \pi((\beta_u)_{|\alpha})((\beta_v)_{|\alpha}) \left(\bigotimes_{j=1}^{r+s-2} (\beta_{f_{u,v}(j)})_{|\alpha} \right).$$

This tool is used in Section 1.5 to obtain the extended combinatorial Ricci curvature tensor Ric from the Riemannian curvature tensor \mathcal{R} . The Riemannian curvature tensor is a $(3, 1)$ -tensor and the extended combinatorial Ricci curvature tensor is a $(2, 0)$ -tensor defined as $C^{2,4}\mathcal{R}$.

Functions and tensor fields: In the context of this thesis, a function h on a CW-complex is a p -chain $h = \sum \lambda_k \beta_k$ (or a p -cochain since both spaces are identified via the inner product g), since we assign to each p -cell β_k a number λ_k . We can view the function h as a constant tensor field $T(h)$ of type $(1, 0)$ (or of type $(0, 1)$ by lowering the index) that is defined cellwise by $T(h)_{|\beta} := h$. A $(1, 0)$ -tensor field will also be called functions with values in $C_p(M; \mathbb{F})$.

Similarly, we can view $T(h)$ as a function $F(T(h))$. Or even more generally, we can define a function associated to an arbitrary tensor field X of type $(1, 0)$. We define

$$F(X) := \sum_{v \in f_p} \left(\sum_{u \in f_p} (X_{|\beta_v})_u \right) \beta_v.$$

This definition does not imply $F(T(h)) = h$.

1.2 AN EXAMPLE: THE 3-DIMENSIONAL CUBE

We now discuss an example to motivate and illustrate the constructions of the following sections. The boundary of the three-dimensional cube shown in Figure 1.2 is a small but nevertheless interesting closed surface. The combinatorics of this object is fairly easy: Every

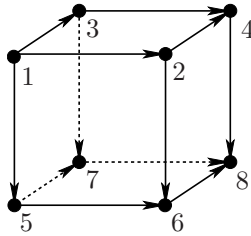


Figure 1.2: An oriented three-dimensional cube. The edges are oriented such that the smaller vertex label is the tail and the larger vertex label is the head of the arrow. 2-faces are oriented such that the lexicographically smallest edge in the boundary has positive relative orientation.

vertex is of degree three and every edge has precisely two parallel neighbours of type \parallel^γ and four transverse neighbours. No edge has a parallel neighbour of type \parallel_α . To compute the matrices associated to the Hodge-Laplacian Δ and the Bochner-Laplacian Δ^∇ for 1-cells, we have to specify an ordering of the 12 edges and an orientation of the edges and 2-faces: An edge is represented by a pair (x, y) of two numbers $x, y \in \{1, \dots, 8\}$ with $x < y$ that denote its endpoints. This convention orients the edges. We order the edges lexicographically, i.e. $(1, 2)$ is the first and $(7, 8)$ the twelfth edge. Each 2-face γ is oriented such that $[\gamma : \beta] = 1$ for the smallest edge β in its boundary. The object β_j refers to the j^{th} edge according to this ordering, e.g. $\beta_5 = (2, 6)$. We choose the geometric set of weights, that is, every vertex α is assigned the weight $w_\alpha = \frac{1}{\sqrt{3}}$, every edge β is assigned the weight $w_\beta = 1$, and every 2-cell γ is assigned the weight $w_\gamma = \sqrt{4} = 2$. It is now an easy task to compute (with the help of Equation 1.4) the (first) Hodge-Laplacian Δ :

$$\Delta = \begin{pmatrix} \frac{7}{6} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{12} & \frac{7}{6} & \frac{1}{12} & -\frac{1}{4} & 0 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ \frac{1}{12} & \frac{1}{12} & \frac{7}{6} & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{4} & 0 & \frac{7}{6} & \frac{1}{12} & \frac{1}{12} & 0 & -\frac{1}{12} & 0 & 0 & -\frac{1}{4} & 0 \\ -\frac{1}{12} & 0 & -\frac{1}{4} & \frac{1}{12} & \frac{7}{6} & 0 & 0 & -\frac{1}{4} & \frac{1}{12} & 0 & -\frac{1}{12} & 0 \\ -\frac{1}{4} & -\frac{1}{12} & 0 & \frac{1}{12} & 0 & \frac{7}{6} & \frac{1}{12} & -\frac{1}{12} & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{12} & -\frac{1}{4} & 0 & 0 & \frac{1}{12} & \frac{7}{6} & -\frac{1}{4} & 0 & \frac{1}{12} & 0 & -\frac{1}{12} \\ 0 & 0 & 0 & -\frac{1}{12} & -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{4} & \frac{7}{6} & 0 & 0 & \frac{1}{12} & \frac{1}{12} \\ -\frac{1}{4} & 0 & -\frac{1}{12} & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{7}{6} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & \frac{7}{6} & -\frac{1}{4} & -\frac{1}{12} \\ 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{12} & 0 & 0 & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{4} & \frac{7}{6} & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & -\frac{1}{12} & \frac{1}{12} & \frac{7}{6} \end{pmatrix}.$$

In this specific example β_j and β_k are transverse neighbours if and only if $\Delta_{jk} = -\frac{1}{4}$. For different choices of weights the entry corresponding to transverse neighbours may vanish; this happens for example if one chooses a *standard set of weights*.

After computing the first Hodge-Laplacian Δ , we want to construct a combinatorial analogue of the rough Laplacian Δ^∇ that we call *Bochner-Laplacian*. In differential geometry, this operator is defined by the covariant derivative ∇ and its adjoint operator ∇^* :

$$\Delta^\nabla := \nabla^* \nabla.$$

To mimick this construction, we first aim for a combinatorial analogue of the covariant derivative ∇ which we call a *combinatorial difference operator*. For precise definitions of ∇ we refer to Section 1.3. We recall that we need some workaround for a technical problem. In Weitzenböck's formula of differential geometry, the rough Laplacian and the Ricci curvature are $(1, 1)$ -tensor fields. This is also true in the combinatorial setting. But the combinatorial Hodge-Laplacian is not a $(1, 1)$ -tensor, it maps functions to functions. The answer to this problem is as follows (and already mentioned in the introduction): We view a p -chain h as a constant $(1, 0)$ -tensor $T(h)$ as described in the previous section. Then we consider and apply fibrewise the diagonal part of the extended Bochner-Laplacian and of the extended combinatorial Ricci curvature. The resulting $(1, 0)$ -tensors are then translated into functions according to the preceding section.

As the covariant derivative of a smooth manifold changes from point to point, the *combinatorial difference operator* ∇ varies from cell to cell. We therefore describe the difference operator $\nabla|_{\beta_k}$ at edge β_k . The combinatorial difference operator at edge β_k again is obtained from *difference mappings* $(D_{\beta_j})|_{\beta_k}$ at β_k in direction of β_j at β_k by linear extension. The difference mappings remind one of a directional derivative, where the non-vanishing combinatorial directions are all neighbouring edges of β_k . In our example, the first edge $\beta_1 = (1, 2)$ has parallel neighbours $\beta_6 = (3, 4)$ and $\beta_9 = (5, 6)$ and transverse neighbours $\beta_2 = (1, 3)$, $\beta_3 = (1, 5)$, $\beta_4 = (2, 4)$, and $\beta_5 = (2, 6)$; we therefore have the following non-zero combinatorial difference mappings at β_1 :

$$(D_{\beta_2})|_{\beta_1}, \quad (D_{\beta_3})|_{\beta_1}, \quad (D_{\beta_4})|_{\beta_1}, \quad (D_{\beta_5})|_{\beta_1}, \quad (D_{\beta_6})|_{\beta_1}, \quad (D_{\beta_9})|_{\beta_1}.$$

A difference mapping $(D_{\beta_j})|_{\beta_k}$ is a 12×12 -matrix of which at least 10 columns and rows are zero: all but possibly rows j and k and columns j and k vanish. We denote these matrices in a reduced form as 2×2 -matrices by deleting all rows and columns except for rows and columns j and k . This reduction is indicated by “ \doteq ”. To compute these difference mappings according to Definition 1.3.1, we have to determine some coefficients that transform the matrices properly if one changes orientations of cells. These coefficients are $\sigma_{\alpha,jk} := \sqrt{[\beta_j : \alpha][\beta_k : \alpha]}$ and $\sigma_{\gamma,jk} := \sqrt{[\gamma : \beta_j][\gamma : \beta_k]}$. For the parallel neighbours β_6 and β_9 of β_1 we have

$$\sigma_{\gamma,6\ 1} = \sqrt{[\gamma : \beta_6][\gamma : \beta_1]} = \sqrt{(-1) \cdot 1} = i, \text{ and } \sigma_{\gamma,9\ 1} = \sqrt{[\gamma : \beta_9][\gamma : \beta_1]} = i.$$

The definition or Lemma 1.3.3 yields

$$(D_{\beta_6})|_{\beta_1} \doteq \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{4}} & -\frac{i}{\sqrt{4}} \end{pmatrix} \quad \text{and} \quad (D_{\beta_9})|_{\beta_1} \doteq \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{4}} & -\frac{i}{\sqrt{4}} \end{pmatrix}.$$

For the transverse neighbours of β_1 we obtain by similar computations

$$\begin{aligned} \sigma_{\alpha,2\ 1} &= 1, & \sigma_{\alpha,3\ 1} &= 1, & \sigma_{\alpha,4\ 1} &= i, & \sigma_{\alpha,5\ 1} &= i, \\ \sigma_{\gamma,2\ 1} &= i, & \sigma_{\gamma,3\ 1} &= i, & \sigma_{\gamma,4\ 1} &= 1, & \sigma_{\gamma,5\ 1} &= 1. \end{aligned}$$

Moreover, we have to determine another coefficient $\tau_{j\ 1}$ that is defined in Section 1.3. To this end, we first observe that the number $n_{\alpha,1}$ of transverse neighbours of edge β_1 via any of its endpoints is 2, that is, $n_{\alpha,1} = 2$, and that the number $n_{\gamma,1}$ of transverse neighbours of edge β_1 via any of its two incident 2-cells is 2, that is, $n_{\gamma,1} = 2$. In our example, these coefficients are the same for all edges. Hence

$$\tau_{2\ 1} = i\sqrt{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4}} = \frac{i}{2}\sqrt{\frac{7}{6}}, \quad \tau_{3\ 1} = \frac{i}{2}\sqrt{\frac{7}{6}}, \quad \tau_{4\ 1} = -\frac{1}{2}\sqrt{\frac{7}{6}}, \quad \text{and} \quad \tau_{5\ 1} = -\frac{1}{2}\sqrt{\frac{7}{6}},$$

and we have (again by using the definition or by Lemma 1.3.4) the following difference

mappings in transverse direction at β_1 :

$$\begin{aligned} (D_{\beta_2})|_{\beta_1} &\doteq \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{7}{6}} & \frac{1}{\sqrt{3}} \\ \frac{i}{2}\sqrt{\frac{7}{6}} & -\frac{i}{\sqrt{4}} \end{pmatrix}, & (D_{\beta_3})|_{\beta_1} &\doteq \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{7}{6}} & \frac{1}{\sqrt{3}} \\ \frac{i}{2}\sqrt{\frac{7}{6}} & -\frac{i}{\sqrt{4}} \end{pmatrix}, \\ (D_{\beta_4})|_{\beta_1} &\doteq \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{7}{6}} & \frac{i}{\sqrt{3}} \\ -\frac{i}{2}\sqrt{\frac{7}{6}} & -\frac{1}{\sqrt{4}} \end{pmatrix}, & \text{and } (D_{\beta_5})|_{\beta_1} &\doteq \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{7}{6}} & \frac{i}{\sqrt{3}} \\ -\frac{i}{2}\sqrt{\frac{7}{6}} & -\frac{1}{\sqrt{4}} \end{pmatrix}. \end{aligned}$$

According to Definition 1.3.5, the difference operator $\nabla|_{\beta_1}$ at β_1 is therefore

$$\nabla|_{\beta_1} = \sum_{j=2}^6 \beta^j \otimes (D_{\beta_j})|_{\beta_1} + \beta^9 \otimes (D_{\beta_9})|_{\beta_1}.$$

Section 1.4 is devoted to the computation of the (extended) Bochner-Laplacian $\tilde{\Delta}|_{\beta_1}^\nabla$ at β_1 , that is, $\nabla|_{\beta_1}^* \nabla|_{\beta_1}$. We still lack the adjoint operator $\nabla|_{\beta_1}^*$ at β_1 of $\nabla|_{\beta_1}$ but by the definition of the inner product on $C^1(M; \mathbb{F}) \otimes C_1(M; \mathbb{F})$, the adjoint operator is

$$\nabla|_{\beta_1}^* = \sum_{j=2}^6 \beta_j \otimes (D_{\beta_j})|_{\beta_1}^\top + \beta_9 \otimes (D_{\beta_9})|_{\beta_1}^\top,$$

where A^\top denotes the transpose of the matrix A . Altogether, the (extended) Bochner-Laplacian $\tilde{\Delta}|_{\beta_1}^\nabla$ at β_1 computes as

$$\tilde{\Delta}|_{\beta_1}^\nabla = \nabla|_{\beta_1}^* \nabla|_{\beta_1} = \sum_{j \in f_p} (D_{\beta_j})|_{\beta_1}^\top (D_{\beta_j})|_{\beta_1} = \sum_{j=2}^6 (D_{\beta_j})|_{\beta_1}^\top (D_{\beta_j})|_{\beta_1} + (D_{\beta_9})|_{\beta_1}^\top (D_{\beta_9})|_{\beta_1}. \quad (1.5)$$

We can either compute the matrices of this sum directly or read off the entries from Lemma 1.4.2, where these computations were done in general. Here, we first compute two terms directly for one parallel and one transverse neighbour and then relate the results to the statement of Lemma 1.4.2 after we did the calculation. Firstly, consider the term of Equation 1.5 computed from the parallel neighbour β_9 of β_1 :

$$(D_{\beta_9})|_{\beta_1}^\top (D_{\beta_9})|_{\beta_1} \doteq \begin{pmatrix} 0 & \frac{1}{\sqrt{4}} \\ 0 & -\frac{i}{\sqrt{4}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{4}} & -\frac{i}{\sqrt{4}} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{i}{4} \\ -\frac{i}{4} & -\frac{1}{4} \end{pmatrix}.$$

Secondly, we consider the term of Equation 1.5 computed from the transverse neighbour β_3 of β_1 :

$$\begin{aligned} (D_{\beta_3})|_{\beta_1}^\top (D_{\beta_3})|_{\beta_1} &\doteq \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{7}{6}} & \frac{i}{2}\sqrt{\frac{7}{6}} \\ \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{4}} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{7}{6}} & \frac{1}{\sqrt{3}} \\ \frac{i}{2}\sqrt{\frac{7}{6}} & -\frac{i}{\sqrt{4}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{\frac{7}{6}}(\frac{1}{\sqrt{3}} - \frac{1}{2}) \\ -\frac{1}{2}\sqrt{\frac{7}{6}}(\frac{1}{\sqrt{3}} - \frac{1}{2}) & \frac{1}{12} \end{pmatrix} \end{aligned}$$

Recall that these matrices represent in fact 12×12 -matrices. Their entries could alternatively be read off from Lemma 1.4.2 as follows: β_9 is a parallel neighbour of type \parallel^γ of β_1 , so Lemma 1.4.2 (2) says how row 9 of $(\tilde{\Delta}^\nabla)_{|\beta_1}$ looks like. The only non-zero entries are

$$((\tilde{\Delta}^\nabla)_{|\beta_1})_{9 \ 1} = ((D_{\beta_9})_{|\beta_1}^\top (D_{\beta_9})_{|\beta_1})_{9 \ 1} = -\sigma_{\gamma,9 \ 1} \frac{w_{1,1}w_{1,9}}{w_{2,\gamma}^2} = -i\frac{1}{4} = -\frac{i}{4}$$

and

$$((\tilde{\Delta}^\nabla)_{|\beta_1})_{9 \ 9} = ((D_{\beta_9})_{|\beta_1}^\top (D_{\beta_9})_{|\beta_1})_{9 \ 9} = \sigma_{\gamma,9 \ 1}^2 \frac{w_{1,1}w_{1,9}}{w_{2,\gamma}^2} = -\frac{1}{4},$$

where $w_{2,\gamma}$ denotes the weight of the 2-cell γ and $w_{1,j} = 1$ is the weight of edge β_j . Since β_3 is a transverse neighbour of β_1 , we know from Lemma 1.4.2 (3) how row 3 of $(\tilde{\Delta}^\nabla)_{|\beta_1}$ looks like. The only non-zero entries are

$$\begin{aligned} ((\tilde{\Delta}^\nabla)_{|\beta_1})_{3 \ 1} &= ((D_{\beta_3})_{|\beta_1}^\top (D_{\beta_3})_{|\beta_1})_{3 \ 1} \\ &= \tau_{3 \ 1} \sigma_{\alpha,3 \ 1} \sigma_{\gamma,3 \ 1} \left(\frac{w_{0,\alpha}}{\sqrt{w_{1,1}w_{1,3}}} - \frac{\sqrt{w_{1,1}w_{1,3}}}{w_{2,\gamma}} \right) = -\frac{1}{2} \sqrt{\frac{7}{6}} \left(\frac{1}{\sqrt{3}} - \frac{1}{2} \right) \end{aligned}$$

and

$$\begin{aligned} ((\tilde{\Delta}^\nabla)_{|\beta_1})_{3 \ 3} &= ((D_{\beta_3})_{|\beta_1}^\top (D_{\beta_3})_{|\beta_1})_{3 \ 3} \\ &= \sigma_{\alpha,3 \ 1}^2 \frac{w_{0,\alpha}}{w_{1,1}w_{1,3}} + \sigma_{\gamma,3 \ 1}^2 \frac{w_{1,1}w_{1,3}}{w_{2,\gamma}^2} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \end{aligned}$$

where $w_{2,\gamma}$ and $w_{1,j}$ are as above and $w_{0,\alpha}$ is the weight of the vertex α .

The first row of $(\tilde{\Delta}^\nabla)_{|\beta_1}$ is the sum of the first rows of $(D_{\beta_j})_{|\beta_1}^\top (D_{\beta_j})_{|\beta_1}$ for all neighbouring edges β_j of β_1 . We now explain how the term that comes from $((D_{\beta_j})_{|\beta_1}^\top (D_{\beta_j})_{|\beta_1})$ can be read off from Lemma 1.4.2 (4). This will be explained using the neighbours β_j , $j \in \{3, 9\}$. The transverse neighbour β_3 of β_1 adds

$$\tau_{3 \ 1} \sigma_{\alpha,3 \ 1} \sigma_{\gamma,3 \ 1} \left(\frac{w_{0,\alpha}}{\sqrt{w_{1,1}w_{1,3}}} - \frac{\sqrt{w_{1,1}w_{1,3}}}{w_{2,\gamma}} \right) \beta^3$$

to the first row (this corresponds to the first row of $(D_{\beta_3})_{|\beta_1}^\top (D_{\beta_3})_{|\beta_1}$ as computed above) while the parallel neighbour β_9 of β_1 adds

$$-\sigma_{\gamma,9 \ 1} \frac{w_{1,1}w_{1,9}}{w_{2,\gamma}^2} \beta^9 + \frac{w_{1,1}w_{1,9}}{w_{2,\gamma}^2} \beta^1$$

to the first row (this corresponds to the first row of $(D_{\beta_9})_{|\beta_1}^\top (D_{\beta_9})_{|\beta_1}$ as computed above).

If we set $A = -\frac{1}{2}\sqrt{\frac{7}{6}}(\frac{1}{\sqrt{3}} - \frac{1}{2})$, we obtain for the (extended) Bochner-Laplacian $\tilde{\Delta}_{|\beta_1}^\nabla$ at β_1

$$\tilde{\Delta}_{|\beta_1}^\nabla = \begin{pmatrix} \frac{2}{4} & A & A & -A & -A & -\frac{i}{4} & 0 & 0 & -\frac{i}{4} & 0 & 0 & 0 \\ A & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The diagonal yields the vector ϑ_1^1 as described by Corollary 1.4.3 for general weights or Corollary 1.4.5 for the special case of the geometric set of weights. The diagonals of $\tilde{\Delta}_{|\beta_j}^\nabla$ (or in other words the vectors ϑ_j^1) are used to define the (condensed) Bochner-Laplacian Δ^∇ as follows: Row j of the condensed Bochner-Laplacian is ϑ_j^1 . The (condensed) Bochner-Laplacian Δ^∇ of the three-dimensional cube with geometric set of weights is therefore the following (12×12) -matrix:

$$\Delta^\nabla = \begin{pmatrix} \frac{2}{4} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{12} & \frac{2}{4} & \frac{1}{12} & -\frac{1}{4} & 0 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ \frac{1}{12} & \frac{1}{12} & \frac{2}{4} & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{12} & \frac{1}{12} & 0 & -\frac{1}{12} & 0 & 0 & -\frac{1}{4} & 0 \\ -\frac{1}{12} & 0 & -\frac{1}{4} & \frac{1}{12} & \frac{2}{4} & 0 & 0 & -\frac{1}{4} & \frac{1}{12} & 0 & -\frac{1}{12} & 0 \\ -\frac{1}{4} & -\frac{1}{12} & 0 & \frac{1}{12} & 0 & \frac{2}{4} & \frac{1}{12} & -\frac{1}{12} & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{12} & -\frac{1}{4} & 0 & 0 & \frac{1}{12} & \frac{2}{4} & -\frac{1}{4} & 0 & \frac{1}{12} & 0 & -\frac{1}{12} \\ 0 & 0 & 0 & -\frac{1}{12} & -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{4} & \frac{2}{4} & 0 & 0 & \frac{1}{12} & \frac{1}{12} \\ -\frac{1}{4} & 0 & -\frac{1}{12} & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{2}{4} & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & \frac{2}{4} & -\frac{1}{4} & -\frac{1}{12} \\ 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{12} & 0 & 0 & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{4} & \frac{2}{4} & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & -\frac{1}{12} & \frac{1}{12} & \frac{2}{4} \end{pmatrix}.$$

At this point we make three observations:

1. Each non-diagonal entry $(\Delta^\nabla)_{jk}$ of the (condensed) Bochner-Laplacian coincides with the entry Δ_{jk} of the Hodge-Laplacian.
2. The diagonal entries $(\Delta^\nabla)_{kk}$ of the (condensed) Bochner-Laplacian are given by the sum of the moduli of the entries that correspond to parallel neighbours of β_k , i.e.,

$$(\Delta^\nabla)_{kk} = \sum_{\beta_j \parallel \beta_k} |(\Delta^\nabla)_{jk}| = \sum_{\beta_j \parallel \beta_k} |\Delta_{jk}|$$

3. If weights are chosen such that the entries of the (condensed) Bochner-Laplacian that correspond to transpose neighbours vanish, then the (condensed) Bochner-Laplacian is precisely one of the two matrices postulated by Forman's decomposition of Weitzenböck type.

We now turn to the combinatorial Ricci curvature, details can be found in Section 1.5. The computation of the combinatorial Ricci curvature is inspired by ideas from differential

geometry that will be only sketched in this section. A combinatorial Riemannian curvature $(3, 1)$ -tensor $\mathcal{R}_{|\beta_k}$ at edge β_k can be computed using the difference operator $\nabla_{|\beta_k}$:

$$\mathcal{R}_{|\beta_k}(X, Y)Z := -((\nabla_X \nabla_Y Z)_{|\beta_k} - (\nabla_Y \nabla_X Z)_{|\beta_k} - (\nabla_{[X, Y]} Z)_{|\beta_k}),$$

where X, Y, Z are 1-chains (interpreted as constant $(1, 0)$ -tensor fields) and the Lie-bracket $[X, Y]$ is defined as $\nabla_X Y - \nabla_Y X$. Since we want second order differences to vanish, that is, $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$, we have $\mathcal{R}_{|\beta_k}(X, Y)Z = (\nabla_{[X, Y]} Z)_{|\beta_k}$. The extended combinatorial Ricci curvature $(2, 0)$ -tensor $\widetilde{\text{Ric}}_{|\beta_k}$ at β_k is the trace of $\mathcal{R}_{|\beta_k}$, a $(2, 0)$ -tensor described in detail in Lemma 1.5.7. But we are rather interested in the Ricci curvature as a $(1, 1)$ -tensor. The result of the conversion is stated in Corollary 1.5.8. This (extended) combinatorial Ricci curvature $(1, 1)$ -tensor is then condensed by the same procedure by which the (extended) Bochner-Laplacian has been condensed: The diagonal ρ_k^1 of the extended Ricci curvature $(1, 1)$ -tensor $\widetilde{\text{Ric}}_{|\beta_k}$ becomes column k of the (condensed) Ricci curvature $(1, 1)$ -tensor Ric . It follows from Corollary 1.5.8 and is stated explicitly in Equation 1.6 of Section 1.5 that $(\rho_k^1)_j \neq 0$ may only happen if $j = k$. In our example, this entry is given as

$$(\rho_k^1)_k = - \sum_{\beta_j || \gamma \beta_k} \frac{w_{1,j} w_{1,k}}{w_{2,\gamma}^2} - \sum_{\beta_j \nabla \gamma \beta_k} \tau_j^2 \sigma_{\alpha,j}^2.$$

So for $k = 1$ we obtain

$$\text{Ric}_{1 \ 1} = (\rho_1^1)_1 = -\frac{1}{4} - \frac{1}{4} - \left(-\frac{7}{24} - \frac{7}{24} - \frac{7}{24} - \frac{7}{24} \right) = -\frac{2}{4} + \frac{7}{6} = \frac{2}{3}.$$

This is the combinatorial Ricci curvature of the first edge of the cube according to our ordering of the edges. By symmetry, the Ricci curvature is the same for all edges of our example:

$$\text{Ric} = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}.$$

In Chapter 2 we give some applications. The first is a proof of a formula of Weitzenböck-type. We have already computed all objects which occur in Weitzenböck's formula and observe for the special case of our example:

$$\Delta = \Delta^\nabla + \text{Ric}.$$

This equation is proved in Section 2.1 as Theorem 2.1.1. If the weights are chosen such that the entries of the Hodge-Laplacian which correspond to transverse neighbours vanish, e.g., by choosing a *standard set of weights*, this equation is precisely the Weitzenböck decomposition Forman postulated to define the combinatorial Ricci curvature.

In Section 2.2 we prove a combinatorial analogue of the famous theorem of Gauß and Bonnet for closed cellular surfaces. The classical theorem relates the Euler number $\chi(S)$ of the surface S to the integral over S of the Ricci curvature: $\int_S \text{Ric} = 2\pi\chi(S)$. In the combinatorial analogue, we choose the geometric set of weights and replace the integral by summing the combinatorial Ricci curvatures of all edges:

$$\sum_{e \text{ edge}} \text{Ric}(e) = 4\chi(S).$$

Theorem 2.2.1 yields this equation for every closed combinatorial 2-manifold weighted by the geometric set of weights. In our example we have

$$\sum_{e \text{ edge}} \text{Ric}(e) = 12 \cdot \frac{2}{3} = 4 \cdot 2 = 4\chi(3\text{-cube}).$$

The choice of weights is very important for this theorem as it does not hold for a standard set of weights. Forman proved in this case that every closed combinatorial surface admits a triangulation such that every edge has negative combinatorial Ricci curvature.

1.3 THE DIFFERENCE OPERATOR

Definitions 1.3.1 and 1.3.5 are the key to all subsequent constructions. As explained in the introduction, we consider the combinatorial difference operator as an analogue of the covariant derivative and force the second order differences to commute. To avoid confusion, we point out that this definition can be interpreted in two different ways: The difference operator can be applied to functions and $(1, 0)$ -tensors. If we apply the difference operator to a function f , that is, a p -(co)chain, the result at a cell β_k depends on local data of the function at β_k , that is, the values $f(\beta)$ have influence on the result $(\nabla_X f)|_{\beta_k}$ if β is a neighbour of β_k . If we consider a $(1, 0)$ -tensor, we do all computation fibrewise and have no influence of local data. We use the difference operator as a (fibrewise linear) operator on $(1, 0)$ -tensors only. Its definition is stated fibrewise.

The change of the coefficient ring in the following definition might seem a bit odd, but it will turn out to be an auxiliary construction. The definition is stated as general as possible (with respect to possible weights). If one restricts to special classes of weights, the terms often simplify significantly. Before we present Definition 1.3.1, we have to introduce some constants. Let us fix a p -cell β_k . The constants $\sigma_{\alpha, jk}$ and $\sigma_{\gamma, jk}$ describe how the orientations of β_k and β_j relate taking the “connecting” cell α or γ into account. They are necessary because we want to control how the difference operator behaves with respect to reorientation of cells. By $n_{\alpha, k}$ and $n_{\gamma, k}$ we count the neighbours of β_k that are transverse

to β_k via α or γ .

$$\begin{aligned}\sigma_{\alpha,jk} &:= \sqrt{[\beta_j : \alpha][\beta_k : \alpha]} \in \{1; i\} \text{ for } \beta_k \cap \beta_j = \alpha \text{ and } \alpha \in K_{p-1} \\ \sigma_{\gamma,jk} &:= \sqrt{[\gamma : \beta_j][\gamma : \beta_k]} \in \{1; i\} \text{ for } \beta_k, \beta_j < \gamma \text{ and } \gamma \in K_{p+1} \\ n_{\alpha,k} &:= |\{\beta_j \in K_p \mid \beta_j \cap_\alpha^\gamma \beta_k \text{ for some } \gamma\}| \\ n_{\gamma,k} &:= |\{\beta_j \in K_p \mid \beta_j \cap_\alpha^\gamma \beta_k \text{ for some } \alpha\}| \\ \tau_{jk} &:= \begin{cases} i\sigma_{\alpha,jk} \sqrt{\frac{1}{n_{\alpha,k}} \cdot \left(\frac{w_{(p-1),\alpha}}{w_{p,k}}\right)^2 + \frac{1}{n_{\gamma,k}} \cdot \left(\frac{w_{p,k}}{w_{(p+1),\gamma}}\right)^2} & \beta_j \cap_\alpha^\gamma \beta_k, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

We emphasise the fact that in general $\tau_{jk} \neq \tau_{kj}$. Recall that all mappings are given with respect to the orthonormal basis β_j defined in Section 1.1.

Definition 1.3.1. The p^{th} difference mapping $(D_{\beta_j}^p)_{|\beta_k} : C_p(M; \mathbb{F}) \longrightarrow C_p(M; \mathbb{C})$ at the p -cell β_k in direction of the p -cell β_j is defined by

$$\beta \longmapsto (D_{\beta_j}^p \beta)_{|\beta_k} := \begin{cases} \frac{w_{(p-1),\alpha}}{\sqrt{w_{p,j}w_{p,k}}}(\lambda_k - \sigma_{\alpha,jk}\lambda_j)\beta_j & \text{for } \beta_j \parallel_\alpha \beta_k, \\ \frac{\sqrt{w_{p,j}w_{p,k}}}{w_{(p+1),\gamma}}(\lambda_k - \sigma_{\gamma,jk}\lambda_j)\beta_j & \text{for } \beta_j \parallel^\gamma \beta_k, \\ (\tau_{jk}\sigma_{\gamma,jk}\lambda_k + \sigma_{\alpha,jk}\frac{w_{(p-1),\alpha}}{\sqrt{w_{p,j}w_{p,k}}}\lambda_j)\beta_k \\ \quad + (\tau_{jk}\sigma_{\alpha,jk}\lambda_k - \sigma_{\gamma,jk}\frac{\sqrt{w_{p,j}w_{p,k}}}{w_{(p+1),\gamma}}\lambda_j)\beta_j & \text{for } \beta_j \cap_\alpha^\gamma \beta_k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta = \sum_{r \in [f_p]} \lambda_r \beta_r$.

The difference mapping is \mathbb{F} -linear by definition and the matrix associated to $(D_{\beta_j}^p)_{|\beta_k}$ has non-zero entries in rows and columns j and k . Thus it can be reduced to a (2×2) -matrix by deleting all but column and row j and k . This reduction is indicated by “ \doteq ”. If we assume $k < j$, we obtain the following three types of matrices for $\beta_j \parallel_\alpha \beta_k$, $\beta_j \parallel^\gamma \beta_k$, and $\beta_j \cap_\alpha^\gamma \beta_k$.

Lemma 1.3.2. If β_j is a \parallel_α -neighbour of β_k , then

$$(D_{\beta_j})_{|\beta_k} \doteq \frac{w_{(p-1),\alpha}}{\sqrt{w_{p,j}w_{p,k}}} \begin{pmatrix} 0 & 0 \\ 1 & -\sigma_{\alpha,jk} \end{pmatrix}.$$

Lemma 1.3.3. If β_j is a \parallel^γ -neighbour of β_k , then

$$(D_{\beta_j})_{|\beta_k} \doteq \frac{\sqrt{w_{p,j}w_{p,k}}}{w_{(p+1),\gamma}} \begin{pmatrix} 0 & 0 \\ 1 & -\sigma_{\gamma,jk} \end{pmatrix}.$$

Lemma 1.3.4. If β_j is a $\mathfrak{M}_\alpha^\gamma$ -neighbour of β_k , then

$$(D_{\beta_j})|_{\beta_k} \doteq \begin{pmatrix} \tau_{jk}\sigma_{\gamma,jk} & \frac{w_{(p-1),\alpha}}{\sqrt{w_{p,j}w_{p,k}}}\sigma_{\alpha,jk} \\ \tau_{jk}\sigma_{\alpha,jk} & -\frac{\sqrt{w_{p,j}w_{p,k}}}{w_{(p+1),\gamma}}\sigma_{\gamma,jk} \end{pmatrix}.$$

We now extend the difference mapping to arbitrary “ $C_p(M; \mathbb{F})$ -directions” to obtain a p^{th} $(2, 1)$ -tensor field, the *difference operator*.

Definition 1.3.5. The p^{th} difference operator $\nabla|_{\beta_k} : C_p(M; \mathbb{F}) \otimes C_p(M; \mathbb{F}) \longrightarrow C_p(M; \mathbb{C})$ at the p -cell β_k is defined as

$$(\nabla_{\sum \mu_j \beta_j} \beta)|_{\beta_k} := \sum_{j \in [f_p]} \mu_j \cdot (D_{\beta_j}^p \beta)|_{\beta_k},$$

where $\beta \in C_p(M; \mathbb{F})$.

Fix a p^{th} tensor field X of type $(1, 0)$, a function f , and a p -cell β . As described earlier, it makes sense to compute $(\nabla_{X|_\beta} f)|_\beta$ as well as $(\nabla_{X|_\beta} (T(f))|_\beta)|_\beta$. For functions, we have therefore two different ways to compute differences. We may consider f as a function or a constant tensor field $T(f)$, as described in Section 1.1. In this thesis, we apply the difference operator to $(1, 0)$ -tensors only. Hence, if we write $(\nabla_X f)|_\beta$, we use this as a shorthand of $(\nabla_{X|_\beta} T(f)|_\beta)|_\beta$. Thus no ambiguities should occur. The $(1, 0)$ -tensor $(\nabla_{X|_\beta} T(f)|_\beta)|_\beta$ does not depend on local data of $T(f)$ around β . But if we transform this $(1, 0)$ -tensor $(\nabla_X f)|_\beta$ back a function, the value of this function at β_k depends on the values of f in a neighbourhood of β .

Although one could consider the cellwise product of a p -chain with a p^{th} tensor field, a combinatorial analogue of a Leibniz rule does not hold.

For *second order differences* we want to ensure that the derivatives commute. We agree on the following convention. Let $\beta, X = \sum X_j \beta_j, Y = \sum Y_j \beta_j \in C_p(M; \mathbb{F})$. The first-order difference of β in direction of X at β_k is given by

$$\gamma|_{\beta_k} := (\nabla_X \beta)|_{\beta_k} = \sum_{j \in [f_p]} X_j \cdot (D_{\beta_j} \beta)|_{\beta_k}.$$

A naïve approach to compute the second order difference is to use the above definition and to calculate $(\nabla_Y \gamma)|_{\beta_k}$. But this does not yield $(\nabla_Y (\nabla_X \beta)|_{\beta_k})|_{\beta_k} = (\nabla_X (\nabla_Y \beta)|_{\beta_k})|_{\beta_k}$. Instead, we define

$$(\nabla_Y^p \nabla_X^p \beta)|_{\beta_k} := \sum_{r \in [f_p]} \beta^r(X) \beta^r(Y) (D_{\beta_r}^p (D_{\beta_r}^p \beta)|_{\beta_k})|_{\beta_k}.$$

This definition is the appropriate one: Later, when we *define* a Lie-bracket using the difference operator $[X, Y] := \nabla_X Y - \nabla_Y X$, we are able to capture Jacobi’s identity for the bracket. This identity does not hold if we would use the naïve second order difference described above. Recollect from differential geometry that a connection is *symmetric* if its torsion vanishes, that is, if $\nabla_X Y - \nabla_Y X = [X, Y]$. The convention for second order differences makes the combinatorial difference operator symmetric by definition.

1.4 THE BOCHNER-LAPLACIAN

We start this section by computing the adjoint operator of ∇ . Recall the definition of the inner product of $C^p(M; \mathbb{F}) \otimes C^p(M; \mathbb{F})$ from Section 1.1. To determine the adjoint operator of ∇ , we have to compute the adjoint of $\nabla|_{\beta_k} : C_p(M; \mathbb{F}) \otimes C_p(M; \mathbb{F}) \longrightarrow C_p(M; \mathbb{C})$ at every p -cell β_k . Since $\beta_1, \dots, \beta_{f_p}$ forms an orthonormal basis of $C_p(M; \mathbb{F})$ we have to transpose each factor to obtain the adjoint operator. Hence the adjoint operator $(\nabla^p)^*_{|\beta_k}$ of the combinatorial difference operator $(\nabla^p)_{|\beta_k}$ is given by

$$(\nabla^p)^*_{|\beta_k} = \sum_{j \in [f_p]} (\beta^j)^\top \otimes (D_{\beta_j}^p)^\top_{|\beta_k} = \sum_{j \in [f_p]} \beta_j \otimes (D_{\beta_j}^p)^\top_{|\beta_k}.$$

Definition 1.4.1. The p^{th} extended combinatorial Bochner-Laplacian $\tilde{\Delta}_{|\beta_k}^\nabla$ at $\beta_k \in K_p$ is the composition

$$\tilde{\Delta}_{|\beta_k}^\nabla := (\nabla^p)^*_{|\beta_k} (\nabla^p)_{|\beta_k} : C_p(M; \mathbb{F}) \longrightarrow C_p(M; \mathbb{C}).$$

Lemma 1.4.2. The p^{th} extended combinatorial Bochner-Laplacian $\tilde{\Delta}_{|\beta_k}^\nabla$ at β_k is given by

$$\tilde{\Delta}_{|\beta_k}^\nabla = \sum_{\beta_j \parallel \beta_k} (D_{\beta_j}^p)^\top_{|\beta_k} (D_{\beta_j}^p)_{|\beta_k} + \sum_{\beta_j \pitchfork \beta_k} (D_{\beta_j}^p)^\top_{|\beta_k} (D_{\beta_j}^p)_{|\beta_k}.$$

The associated matrix of $\tilde{\Delta}_{|\beta_k}^\nabla$ has the following non-zero rows:

1. **row j for $\beta_j \parallel_\alpha \beta_k$:**

$$\sigma_{\alpha,jk} \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} (\sigma_{\alpha,jk} \beta^j - \beta^k)$$

2. **row j for $\beta_j \parallel^\gamma \beta_k$:**

$$\sigma_{\gamma,jk} \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} (\sigma_{\gamma,jk} \beta^j - \beta^k)$$

3. **row j for $\beta_j \pitchfork_\alpha^\gamma \beta_k$:**

$$\tau_{jk} \sigma_{\alpha,jk} \sigma_{\gamma,jk} \left(\frac{w_{(p-1),\alpha}}{\sqrt{w_{p,j} w_{p,k}}} - \frac{\sqrt{w_{p,j} w_{p,k}}}{w_{(p+1),\gamma}} \right) \beta^k + \left(\sigma_{\alpha,jk}^2 \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} + \sigma_{\gamma,jk}^2 \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} \right) \beta^j$$

4. **row k :**

$$\begin{aligned} & \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} (-\sigma_{\alpha,jk} \beta^j + \beta^k) + \sum_{\beta_j \parallel^\gamma \beta_k} \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} (-\sigma_{\gamma,jk} \beta^j + \beta^k) \\ & + \sum_{\beta_j \pitchfork_\alpha^\gamma \beta_k} \tau_{jk} \sigma_{\alpha,jk} \sigma_{\gamma,jk} \left(\frac{w_{(p-1),\alpha}}{\sqrt{w_{p,j} w_{p,k}}} - \frac{\sqrt{w_{p,j} w_{p,k}}}{w_{(p+1),\gamma}} \right) \beta^j \end{aligned}$$

Proof. Both claims follow from direct computations. We have

$$(\nabla^p)_{|\beta_k}^* (\nabla^p)_{|\beta_k} = (\nabla^p)_{|\beta_k}^* \left(\sum_{r \in [f_p]} \beta^r \otimes (D_{\beta_r}^p)_{|\beta_k} \right) = \sum_{r \in [f_p]} (D_{\beta_r}^p)_{|\beta_k}^\top (D_{\beta_r}^p)_{|\beta_k}.$$

Multiplication of $(D_{\beta_j}^p)_{|\beta_k}^\top$ and $(D_{\beta_j}^p)_{|\beta_k}$ only yields nonzero matrices in case of $\beta_j \parallel \beta_k$ or $\beta_j \cap \beta_k$. The structure of $(D_{\beta_j}^p)_{|\beta_k}$ as stated in Lemma 1.3.2, Lemma 1.3.3, and Lemma 1.3.4 guarantees that the only nonzero entries of $(D_{\beta_j}^p)_{|\beta_k}^\top (D_{\beta_j}^p)_{|\beta_k}$ can occur at positions (j, j) , (k, k) , (j, k) , or (k, j) for $\beta_j \parallel \beta_k$ or $\beta_j \cap \beta_k$. To check that

$$\left((D_{\beta_j}^p)_{|\beta_k}^\top (D_{\beta_j}^p)_{|\beta_k} \right)_{kk} = 0 \quad \text{for } \beta_j \cap_\alpha^\gamma \beta_k$$

one uses Equation 1.2. All corresponding numerical entries are easily computed, and summing these matrices yields the desired result. \square

As mentioned already in Section 1.2, we are chiefly interested in the vector determined by the diagonal entries of the extended combinatorial Bochner-Laplacian $\tilde{\Delta}_{|\beta_k}^\nabla$ at β_k . We interpret this f_p -tuple as an element of \mathbb{F}^{f_p} and denote it by ϑ_k^p . From the preceding Lemma 1.4.2, we get

Corollary 1.4.3. Weigh the cells of a CW-complex by arbitrary positive weights. Then the diagonal of the p^{th} extended combinatorial Bochner-Laplacian at β_k is given by

$$(\vartheta_k^p)_j = \begin{cases} \sigma_{\alpha,jk}^2 \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} & \beta_j \parallel_\alpha \beta_k, \\ \sigma_{\gamma,jk}^2 \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} & \beta_j \parallel_\gamma \beta_k, \\ \sigma_{\alpha,jk}^2 \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} + \sigma_{\gamma,jk}^2 \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} & \beta_j \cap_\alpha^\gamma \beta_k, \\ \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} + \sum_{\beta_j \parallel_\gamma \beta_k} \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} & j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1.4.4. Weigh the cells of a CW-complex M by a standard set of weights, that is, a p -cell is assigned the weight $w_p = \sqrt{\kappa_1 \cdot \kappa_2^p}$. Then the diagonal of the p^{th} extended combinatorial Bochner-Laplacian at β_k is given by

$$(\vartheta_k^p)_j = \begin{cases} \sigma_{\xi,jk}^2 \frac{1}{\kappa_2} & \beta_j \parallel_\xi \beta_k \text{ or } \beta_j \parallel_\xi^\xi \beta_k \\ \sum_{\beta_j \parallel \beta_k} \frac{1}{\kappa_2} & j = k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi \in K_{p+1}$.

Corollary 1.4.5. Weigh the cells of a cellular surface by the geometric set of weights. Then the diagonal of the p^{th} extended combinatorial Bochner-Laplacian at β_k is given by

$$(\vartheta_k^p)_j = \begin{cases} \sigma_{\alpha,jk}^2 \frac{1}{\deg(\alpha)} & \beta_j \parallel_\alpha \beta_k, \\ \sigma_{\gamma,jk}^2 \frac{1}{\text{sides}(\gamma)} & \beta_j \parallel^\gamma \beta_k, \\ \sigma_{\alpha,jk}^2 \frac{1}{\deg(\alpha)} + \sigma_{\gamma,jk}^2 \frac{1}{\text{sides}(\gamma)} & \beta_j \pitchfork_\alpha^\gamma \beta_k, \\ \sum_{\beta_j \parallel_\alpha \beta_k} \frac{1}{\deg(\alpha)} + \sum_{\beta_j \parallel^\gamma \beta_k} \frac{1}{\text{sides}(\gamma)} & j = k, \\ 0 & \text{otherwise.} \end{cases}$$

We end this section with a definition and an observation.

Definition 1.4.6. The p^{th} condensed combinatorial Bochner-Laplace operator

$$\Delta_p^\nabla : C_p(M; \mathbb{F}) \longrightarrow C_p(M; \mathbb{F})$$

is defined by the matrix $\sum_{k \in [f_p]} \beta^k \otimes \vartheta_k^p$, i.e. column j of Δ_p^∇ is given by ϑ_j^p .

The condensed combinatorial Bochner-Laplacian describes on the level of functions how the diagonal part of the extended Bochner-Laplacian looks like *at each cell*. Finally, we repeat an observation that was stated already in Section 1.2, but this time it is more than just an example that fits.

Corollary 1.4.7. The non-diagonal entries of the p^{th} (condensed) combinatorial Bochner-Laplacian Δ_p^∇ and the non-diagonal entries of the p^{th} combinatorial Hodge-Laplacian Δ_p are the same:

$$(\Delta_p^\nabla)_{jk} = (\Delta_p)_{jk} \quad \text{for } j \neq k.$$

The diagonal entry $(\Delta_p^\nabla)_{kk}$ of the p^{th} (condensed) combinatorial Bochner-Laplacian Δ_p^∇ is the sum of the moduli of the entries that correspond to the parallel neighbours of β_k :

$$(\Delta_p^\nabla)_{kk} = \sum_{\beta_j \parallel \beta_k} |(\Delta_p^\nabla)_{jk}| = \sum_{\beta_j \parallel \beta_k} |(\Delta_p)_{jk}|.$$

The last statement shows how Δ^∇ differs from Δ^F in general.

1.5 COMBINATORIAL CURVATURE TENSORS

In this section, we compute a combinatorial Riemannian curvature tensor and its trace, the Ricci curvature tensor, using the difference operator ∇ . In (semi-)Riemannian geometry, the Lie bracket of an n -dimensional smooth manifold M is given in local coordinates by

$$[X, Y] = \sum_{i=1}^n \left[X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right] \frac{\partial}{\partial x^i}$$

and the Riemannian curvature tensor is defined as

$$\mathcal{R}(X, Y)Z := \pm (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z),$$

viewed as a $(3, 1)$ -tensor. The choice of the sign involved varies over the literature. The Ricci curvature tensor $\text{Ric}(X, Y)$ is the trace of the endomorphism $Z \mapsto \mathcal{R}(X, Z)Y$.

Definition 1.5.1. The *combinatorial Lie-bracket* $[X, Y]_{|\beta_k}$ of two p -chains $X = \sum X_j \beta_j$ and $Y = \sum Y_j \beta_j$ at $\beta_k \in K_p$ is defined to be the p -chain given by

$$[X, Y]_{|\beta_k} := (\nabla_X^p Y)_{|\beta_k} - (\nabla_Y^p X)_{|\beta_k} = \sum_{\beta_j \in K_p} X_j \cdot (D_{\beta_j}^p Y)_{|\beta_k} - Y_j \cdot (D_{\beta_j}^p X)_{|\beta_k}.$$

The following Lemma is a useful fact that is derived from a straight-forward computation.

Lemma 1.5.2. The combinatorial Lie-bracket $[X, Y]_{|\beta_k}$ of two p -chains $X = \sum X_j \beta_j$ and $Y = \sum Y_j \beta_j$ is given by

$$\begin{aligned} [X, Y]_{|\beta_k} = & \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1), \alpha}}{\sqrt{w_{p,j} w_{p,k}}} (X_j Y_k - X_k Y_j) \beta_j + \sum_{\beta_j \parallel^\gamma \beta_k} \frac{\sqrt{w_{p,j} w_{p,k}}}{w_{(p+1), \gamma}} (X_j Y_k - X_k Y_j) \beta_j \\ & + \sum_{\beta_j \bowtie_\alpha^\gamma \beta_k} \tau_{jk} \sigma_{\gamma, jk} (X_j Y_k - X_k Y_j) \beta_k + \sum_{\beta_j \bowtie_\alpha^\gamma \beta_k} \tau_{jk} \sigma_{\alpha, jk} (X_j Y_k - X_k Y_j) \beta_j. \end{aligned}$$

The combinatorial Lie-bracket satisfies Jacobi's identity. It is essential for the proof that second order differences commute.

Lemma 1.5.3. The combinatorial Lie-bracket satisfies Jacobi's identity:

$$[[X, Y], Z]_{|\beta_k} + [[Y, Z], X]_{|\beta_k} + [[Z, X], Y]_{|\beta_k} = 0.$$

Proof. To ease notation a bit, we omit the index that indicates the cell where we take differences. We have

$$[[X, Y], Z] = \nabla_{[X, Y]} Z - \nabla_Z [X, Y] = \nabla_{\nabla_X Y - \nabla_Y X} Z - \nabla_Z \nabla_X Y + \nabla_Z \nabla_Y X.$$

Together with the fact that second order differences commute we obtain

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = \nabla_{\nabla_X Y - \nabla_Y X} Z + \nabla_{\nabla_Y Z - \nabla_Z Y} X + \nabla_{\nabla_Z X - \nabla_X Z} Y.$$

We now use Lemma 1.5.2. For $\beta_j \parallel^\alpha \beta$ and obtain for the differences in β_j -direction:

$$\begin{aligned} & \frac{w_{(p-1), \alpha}}{\sqrt{w_{p,j} w_{p,k}}} [\nabla_{(X_j Y_k - X_k Y_j) \beta_j} Z + \nabla_{(Y_j Z_k - Y_k Z_j) \beta_j} X + \nabla_{(Z_j X_k - Z_k X_j) \beta_j} Y] \\ &= \frac{w_{(p-1), \alpha}}{\sqrt{w_{p,j} w_{p,k}}} [(X_j Y_k - X_k Y_j) \nabla_{\beta_j} Z + (Y_j Z_k - Y_k Z_j) \nabla_{\beta_j} X \\ & \quad + (Z_j X_k - Z_k X_j) \nabla_{\beta_j} Y] \\ &= \left(\frac{w_{(p-1), \alpha}}{\sqrt{w_{p,j} w_{p,k}}} \right)^2 [(X_j Y_k - X_k Y_j) (Z_k - \sigma_{\alpha, jk} Z_j) + (Y_j Z_k - Y_k Z_j) (X_k - \sigma_{\alpha, jk} X_j) \\ & \quad + (Z_j X_k - Z_k X_j) (Y_k - \sigma_{\alpha, jk} Y_j)] \beta_j \\ &= 0. \end{aligned}$$

The computations for $\beta_j \parallel^\gamma$ and $\beta_j \cap_\alpha^\gamma \beta$ are similar. \square

Definition 1.5.4. The p^{th} combinatorial Riemannian curvature tensor $\mathcal{R}_{|\beta_k}$ at $\beta_k \in K_p$ is given by

$$\mathcal{R}_{|\beta_k}(X, Y)Z := - \left((\nabla_X^p \nabla_Y^p Z)_{|\beta_k} - (\nabla_Y^p \nabla_X^p Z)_{|\beta_k} - (\nabla_{[X, Y]}^p Z)_{|\beta_k} \right),$$

for p -chains X, Y , and Z .

Since second order differences commute, we have $\mathcal{R}_{|\beta_k}(X, Y)Z = (\nabla_{[X, Y]}^p Z)_{|\beta_k}$.

Definition 1.5.5. The p^{th} extended combinatorial Ricci curvature tensor $\widetilde{\text{Ric}}_{|\beta_k}$ at β_k is defined as a trace of the combinatorial Riemannian curvature tensor \mathcal{R} :

$$\widetilde{\text{Ric}}_{|\beta_k}(X, Y) := (C^{2,1}\mathcal{R})_{|\beta_k}(X, Y) = \text{tr} \left(Z \mapsto \mathcal{R}_{|\beta_k}(X, Z)Y \right),$$

for $X, Y \in C_p(M; \mathbb{F})$.

We now describe the matrix of the p^{th} combinatorial curvature tensor $\mathcal{R}_{|\beta_k}$ at β_k .

Lemma 1.5.6. Consider functions X, Y, Z with values in $C_p(M; \mathbb{F})$ such that we have $X_{|\beta_k} = \sum X_j \beta_j$, $Y_{|\beta_k} = \sum Y_j \beta_j$, $Z_{|\beta_k} = \sum Z_j \beta_j$. The p^{th} combinatorial Riemannian curvature tensor $\mathcal{R}_{|\beta_k}$ at β_k satisfies

$$\begin{aligned} (\mathcal{R}(X, Y)Z)_{|\beta_k} &= \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1), \alpha}^2}{w_{p,j} w_{p,k}} (X_j Y_k - X_k Y_j) (Z_k - \sigma_{\alpha, jk} Z_j) \beta_j \\ &\quad + \sum_{\beta_j \parallel_\gamma \beta_k} \frac{w_{p,j} w_{p,k}}{w_{(p+1), \gamma}^2} (X_j Y_k - X_k Y_j) (Z_k - \sigma_{\gamma, jk} Z_j) \beta_j \\ &\quad + \sum_{\beta_j \cap_\alpha^\gamma \beta_k} \tau_{jk} \sigma_{\alpha, jk} (X_j Y_k - X_k Y_j) \left(\tau_{jk} \sigma_{\gamma, jk} Z_k + \frac{w_{(p-1), \alpha}}{\sqrt{w_{p,j} w_{p,k}}} \sigma_{\alpha, jk} Z_j \right) \beta_k \\ &\quad + \sum_{\beta_j \cap_\alpha^\gamma \beta_k} \tau_{jk} \sigma_{\alpha, jk} (X_j Y_k - X_k Y_j) \left(\tau_{jk} \sigma_{\alpha, jk} Z_k - \frac{\sqrt{w_{p,j} w_{p,k}}}{w_{(p+1), \gamma}} \sigma_{\gamma, jk} Z_j \right) \beta_j. \end{aligned}$$

Proof. We only have to compute $(\nabla_{[X, Y]}^p Z)_{|\beta_k}$, since $(\nabla_X^p \nabla_Y^p Z)_{|\beta_k} = (\nabla_Y^p \nabla_X^p Z)_{|\beta_k}$. We plug

the result of Lemma 1.5.2 into the definition of the Riemannian curvature tensor and obtain

$$\begin{aligned}
(\mathcal{R}(X, Y)Z)|_{\beta_k} &= (\nabla_{\sum_{\beta_j \parallel_{\alpha} \beta_k} \frac{w_{(p-1), \alpha}}{\sqrt{w_{p,j} w_{p,k}}} (X_j Y_k - X_k Y_j) \beta_j} Z)|_{\beta_k} \\
&\quad + (\nabla_{\sum_{\beta_j \parallel^{\gamma} \beta_k} \frac{\sqrt{w_{p,j} w_{p,k}}}{w_{(p+1), \gamma}} (X_j Y_k - X_k Y_j) \beta_j} Z)|_{\beta_k} \\
&\quad + (\nabla_{\sum_{\beta_j \pitchfork_{\alpha}^{\gamma} \beta_k} \tau_{jk} \sigma_{\alpha, jk} (X_j Y_k - X_k Y_j) \beta_j} Z)|_{\beta_k} \\
&= \sum_{\beta_j \parallel_{\alpha} \beta_k} \frac{w_{(p-1), \alpha}}{\sqrt{w_{p,j} w_{p,k}}} (X_j Y_k - X_k Y_j) (\nabla_{\beta_j} Z)|_{\beta_k} \\
&\quad + \sum_{\beta_j \parallel^{\gamma} \beta_k} \frac{\sqrt{w_{p,j} w_{p,k}}}{w_{(p+1), \gamma}} (X_j Y_k - X_k Y_j) (\nabla_{\beta_j} Z)|_{\beta_k} \\
&\quad + \sum_{\beta_j \pitchfork_{\alpha}^{\gamma} \beta_k} \tau_{jk} \sigma_{\alpha, jk} (X_j Y_k - X_k Y_j) (\nabla_{\beta_j} Z)|_{\beta_k}.
\end{aligned}$$

The claim follows after we substitute $(\nabla_{\beta_j} Z)|_{\beta_k} = (D_{\beta_j}^p Z)|_{\beta_k}$ into all three sums according to Definition 1.3.1. \square

We now compute the p^{th} extended combinatorial Ricci curvature tensor. We state the Ricci curvature in two equivalent forms. The first one, given by Lemma 1.5.7, looks at $\widetilde{\text{Ric}}|_{\beta_k}$ as a $(2, 0)$ -tensor, that is, for each p -cell β_k the input are two p -chains X and Y and the output is the number $\widetilde{\text{Ric}}|_{\beta_k}(X, Y)$. The second form is as $(1, 1)$ -tensor, that is, at every p -cell β_k a p -chain $\widetilde{\text{Ric}}|_{\beta_k}(X)$ is assigned to a given p -chain X . The reason behind this is our aim to derive a combinatorial analogue of Weitzenböck's formula. It turns out that we have to interpret the Ricci curvature tensor as a $(1, 1)$ -tensor, as one does for smooth manifolds. After we obtained in Corollary 1.5.8 an extended combinatorial Ricci curvature $(1, 1)$ -tensor, we still have to condense it the same way we condensed the Bochner-Laplacian.

Lemma 1.5.7. The p^{th} extended combinatorial Ricci curvature tensor at β_k is given by

$$\begin{aligned}
\widetilde{\text{Ric}}|_{\beta_k}(X, Y) &= - \sum_{\beta_j \parallel_{\alpha} \beta_k} \frac{w_{(p-1), \alpha}^2}{w_{p,j} w_{p,k}} X_k (Y_k - \sigma_{\alpha, jk} Y_j) - \sum_{\beta_j \parallel^{\gamma} \beta_k} \frac{w_{p,j} w_{p,k}}{w_{(p+1), \gamma}^2} X_k (Y_k - \sigma_{\gamma, jk} Y_j) \\
&\quad - \sum_{\beta_j \pitchfork_{\alpha}^{\gamma} \beta_k} \tau_{jk} \sigma_{\alpha, jk} X_k \left(\tau_{jk} \sigma_{\alpha, jk} Y_k + \frac{\sqrt{w_{p,j} w_{p,k}}}{w_{(p+1), \gamma}} \sigma_{\gamma, jk} Y_j \right).
\end{aligned}$$

Proof. By definition we have

$$\begin{aligned}
\widetilde{\text{Ric}}_{|\beta_k}(X, Y) &= \sum_{j \in [f_p] \setminus \{k\}} \beta^j (\mathcal{R}_{|\beta_k}(X, \beta_j)Y) \\
&= \sum_{\beta_j \parallel_\alpha \beta_k} \beta^j \left(\frac{w_{(p-1),\alpha}^2}{w_{p,j}w_{p,k}} (-X_k)(Y_k - \sigma_{\alpha,jk}Y_j) \beta_j \right) \\
&\quad + \sum_{\beta_j \parallel^\gamma \beta_k} \beta^j \left(\frac{w_{p,j}w_{p,k}}{w_{(p+1),\gamma}^2} (-X_k)(Y_k - \sigma_{\gamma,jk}Y_j) \beta_j \right) \\
&\quad + \sum_{\beta_j \pitchfork_\alpha^\gamma \beta_k} \beta^j \left(\tau_{jk}\sigma_{\alpha,jk}(-X_k) \left(\tau_{jk}\sigma_{\gamma,jk}Y_k + \frac{w_{(p-1),\alpha}}{\sqrt{w_{p,j}w_{p,k}}} \sigma_{\alpha,jk}Y_j \right) \beta_k \right) \\
&\quad + \sum_{\beta_j \pitchfork_\alpha^\gamma \beta_k} \beta^j \left(\tau_{jk}\sigma_{\alpha,jk}(-X_k) \left(\tau_{jk}\sigma_{\alpha,jk}Y_k - \frac{\sqrt{w_{p,j}w_{p,k}}}{w_{(p+1),\gamma}} \sigma_{\gamma,jk}Y_j \right) \beta_j \right) \\
&= - \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,j}w_{p,k}} X_k (Y_k - \sigma_{\alpha,jk}Y_j) - \sum_{\beta_j \parallel^\gamma \beta_k} \frac{w_{p,j}w_{p,k}}{w_{(p+1),\gamma}^2} X_k (Y_k - \sigma_{\gamma,jk}Y_j) \\
&\quad - \sum_{\beta_j \pitchfork_\alpha^\gamma \beta_k} \tau_{jk}\sigma_{\alpha,jk}X_k \left(\tau_{jk}\sigma_{\alpha,jk}Y_k - \frac{\sqrt{w_{p,j}w_{p,k}}}{w_{(p+1),\gamma}} \sigma_{\gamma,jk}Y_j \right).
\end{aligned}$$

This yields the claim. \square

Corollary 1.5.8. The p^{th} extended combinatorial Ricci curvature tensor $\widetilde{\text{Ric}}_{|\beta_k}$ viewed as a $(1, 1)$ -tensor is represented by a matrix with one non-zero column. The non-zero column is column k which is given by

$$(\widetilde{\text{Ric}}_{|\beta_k})_{jk} = \begin{cases} \frac{w_{(p-1),\alpha}^2}{w_{p,j}w_{p,k}} \sigma_{\alpha,jk} & \text{if } \beta_j \parallel_\alpha \beta_k, \\ \frac{w_{p,j}w_{p,k}}{w_{(p+1),\gamma}^2} \sigma_{\gamma,jk} & \text{if } \beta_j \parallel^\gamma \beta_k, \\ \tau_{jk}\sigma_{\alpha,jk}\sigma_{\gamma,jk} \frac{\sqrt{w_{p,j}w_{p,k}}}{w_{(p+1),\gamma}} & \text{if } \beta_j \pitchfork_\alpha^\gamma \beta_k, \\ - \sum_{\beta_r \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,r}w_{p,k}} - \sum_{\beta_r \parallel^\gamma \beta_k} \frac{w_{p,r}w_{p,k}}{w_{(p+1),\gamma}^2} - \sum_{\beta_r \pitchfork_\alpha^\gamma \beta_k} \tau_{rk}^2 \sigma_{\alpha,rk}^2 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to the case of the p^{th} extended combinatorial Bochner-Laplacian, we now consider a condensed form of the p^{th} extended combinatorial Ricci curvature tensor that will also be referred to as p^{th} (condensed) combinatorial Ricci curvature tensor Ric . We denote the vector defined by the diagonal entries of the $(1, 1)$ -tensor $\widetilde{\text{Ric}}_{|\beta_k}$ by ρ_k^p . From the

preceeding Corollary 1.5.8 we read off:

$$(\rho_k^p)_j = \begin{cases} -\sum_{\beta_r \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,r} w_{p,k}} - \sum_{\beta_r \parallel^\gamma \beta_k} \frac{w_{p,r} w_{p,k}}{w_{(p+1),\gamma}^2} - \sum_{\beta_r \cap_\alpha^\gamma \beta_k} \tau_{rk}^2 \sigma_{\alpha,rk}^2, & j = k, \\ 0 & \text{otherwise.} \end{cases} \quad (1.6)$$

The diagonal of the p^{th} extended combinatorial Ricci curvature $\widetilde{\text{Ric}}_p$ is now used to define the p^{th} condensed combinatorial Ricci curvature Ric_p that maps p -chains to p -chains.

Definition 1.5.9. The p^{th} (condensed) combinatorial Ricci curvature Ric_p is defined to be the matrix $\sum_{k \in [f_p]} \beta^k \otimes \rho_k^p$.

We recall from the definition of $\sigma_{\alpha,jk}$ and τ_{jk} in Section 1.3 that

$$\tau_{jk}^2 \sigma_{\alpha,jk}^2 = - \left(\frac{1}{n_{\alpha,k}} \cdot \frac{w_{(p-1),\alpha}^2}{w_{p,k}^2} + \frac{1}{n_{\gamma,k}} \cdot \frac{w_{p,k}^2}{w_{(p+1),\gamma}^2} \right).$$

Corollary 1.5.10. The p^{th} (condensed) combinatorial Ricci curvature is a diagonal matrix. If arbitrary positive weights are chosen for the cells of the CW-complex, then we have

$$(\text{Ric}_p)_{kk} = - \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} - \sum_{\beta_j \parallel^\gamma \beta_k} \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} + \sum_{\beta_j \cap_\alpha^\gamma \beta_k} \left(\frac{1}{n_{\alpha,k}} \cdot \frac{w_{(p-1),\alpha}^2}{w_{p,k}^2} + \frac{1}{n_{\gamma,k}} \cdot \frac{w_{p,k}^2}{w_{(p+1),\gamma}^2} \right).$$

Corollary 1.5.11. If we choose a standard set of weights for the CW-complex, that is, we assign the weight $w_p = \sqrt{\kappa_1 \cdot \kappa_2^p}$ to each p -cell, then the p^{th} condensed combinatorial Ricci curvature is a diagonal matrix with

$$(\text{Ric}_p)_{kk} = - \sum_{\beta_j \parallel_\alpha \beta_k} \frac{1}{\kappa_2} + \sum_{\beta_j \cap_\alpha^\gamma \beta_k} \frac{1}{\kappa_2} \left(\frac{1}{n_{\alpha,k}} + \frac{1}{n_{\gamma,k}} \right).$$

Corollary 1.5.12. If we choose the geometric set of weights for a closed cellular surface, then we have

$$(\text{Ric}_1)_{kk} = - \sum_{\beta_j \parallel_\alpha \beta_k} \frac{1}{\deg(\alpha)} - \sum_{\beta_j \parallel^\gamma \beta_k} \frac{1}{\text{sides}(\gamma)} + \sum_{\beta_j \cap_\alpha^\gamma \beta_k} \left(\frac{1}{n_{\alpha,k}} \cdot \frac{1}{\deg \alpha} + \frac{1}{n_{\gamma,k}} \cdot \frac{1}{\text{sides}(\gamma)} \right).$$

CHAPTER 2

APPLICATIONS

As seen in the previous chapter, some fundamental ideas and constructions can be carried over from differential geometry to a purely combinatorial setting of geometry. Sometimes we have to adjust the blueprint here and there, as we did for the *condensed Bochner-Laplacian* and the *condensed Ricci curvature*.

But the abstract concept of a *difference operator*, a *Bochner-Laplacian*, and a *combinatorial Ricci curvature* remain anaemic unless we deduce interesting consequences. Forman pioneered this with his proof of a combinatorial version of Bochner's theorem for 1-chains and of Myers' theorem. His starting point was the *definition* of a formula of Weitzenböck type as described in the introduction. We start this chapter with a *proof* of such a combinatorial formula in Section 2.1. The formula we prove coincides with Forman's formula if and only if the entries of the combinatorial Hodge-Laplacian that correspond to transverse neighbours vanish. This happens for example if we choose a standard set of weights.

In Section 2.2 we study some consequences if we choose the geometric set of weights. Our approach gives for these weights a different notion of Ricci curvature than Forman's definition. The effort we put into a detailed analysis of a weighted variant of the combinatorial Ricci curvature pays off: We are able to prove a combinatorial version of the theorem of Gauß and Bonnet for cellular surfaces weighted by the geometric set of weights. As we argued earlier, such a theorem is impossible for a standard set of weights.

Sections 2.3–2.6 discuss Bochner's theorem in a combinatorial disguise. We summarise Forman's proof of Bochner's theorem for 1-chains in Section 2.3 and scrutinise in Section 2.4 whether this method can be extended to general weights (and our notion of Ricci curvature) or not, by looking at different cell decompositions of a 2-dimensional torus with different sets of weights. Problems arise if one tries to extend Bochner's theorem for 1-chains to p -chains, Forman [26] describes one aspect which we present and discuss briefly in Section 2.5. An important result needed for the proof of Bochner's theorem for 1-chains is a unique continuation theorem: A 1-chain that vanishes *locally* and is contained in $\text{Ker } \delta \cap \text{Ker } \Delta^\nabla$ vanishes globally. Section 2.6 studies problems related to a theorem of this type for 2-chains. Such a theorem is possible if we make an additional assumption. It remains open whether this extra requirement is too restrictive to prove a theorem of Bochner for 2-chains or not.

This chapter closes with Section 2.7 where we analyse possible directions to extend Forman's method to obtain upper bounds for the combinatorial diameter of a positively Ricci curved quasiconvex CW-complex from a standard set of weights to more general weights.

2.1 COMBINATORIAL WEITZENBÖCK FORMULAE

In this section, we relate the p^{th} Hodge-Laplacian, the p^{th} (condensed) combinatorial Bochner-Laplacian, and the p^{th} (condensed) combinatorial Ricci curvature. Since the formula we obtain reminds strongly of the classical formula of Weitzenböck for 1-forms on smooth Riemannian manifolds, we call the formula proven in Theorem 2.1.1 *combinatorial Weitzenböck's formula*, or more precise formulae (since we obtain a formula for each p). The additional condition that the r -skeleton of the CW-complex must be pure is not too restrictive, since the large class of interesting examples given by all combinatorial d -manifolds satisfies this assumption.

Theorem 2.1.1 (combinatorial Weitzenböck's formula).

Let M be a weighted CW-complex with pure r -skeleton and choose $0 < p < r$. Then the following combinatorial Weitzenböck's formula holds:

$$\Delta^p = \Delta_p^\nabla + \text{Ric}_p.$$

If weights are chosen such that the entries of Δ^p that correspond to transverse neighbours vanish, then this formula specialises to the decomposition postulated by Forman [26]:

$$\Delta^p = \Delta_p^F + \text{Ric}_p^F.$$

Proof. From Formula 1.4, Corollary 1.4.7, and Corollary 1.5.10 we immediately read off the claim for all non-diagonal entries. For the diagonal entries it remains to show

$$(\Delta_p)_{kk} = (\Delta_p^\nabla)_{kk} + (\text{Ric}_p)_{kk}.$$

From Corollary 1.4.3 we know

$$(\Delta_p^\nabla)_{kk} = \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} + \sum_{\beta_j \parallel_\gamma \beta_k} \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2},$$

while we have from Corollary 1.5.10 that

$$(\text{Ric}_p)_{kk} = - \sum_{\beta_j \parallel_\alpha \beta_k} \frac{w_{(p-1),\alpha}^2}{w_{p,j} w_{p,k}} - \sum_{\beta_j \parallel_\gamma \beta_k} \frac{w_{p,j} w_{p,k}}{w_{(p+1),\gamma}^2} + \sum_{\beta_j \nparallel_\alpha^\gamma \beta_k} \left(\frac{1}{n_{\alpha,k}} \cdot \frac{w_{(p-1),\alpha}^2}{w_{p,k}^2} + \frac{1}{n_{\gamma,k}} \cdot \frac{w_{p,k}^2}{w_{(p+1),\gamma}^2} \right).$$

Since

$$(\Delta_p)_{kk} = \sum_{\alpha < \beta_k} \left(\frac{w_{(p-1),\alpha}}{w_{p,k}} \right)^2 + \sum_{\gamma > \beta_k} \left(\frac{w_{p,k}}{w_{(p+1),\gamma}} \right)^2,$$

it suffices to show

$$\sum_{\alpha < \beta_k} \left(\frac{w_{(p-1),\alpha}}{w_{p,k}} \right)^2 + \sum_{\gamma > \beta_k} \left(\frac{w_{p,k}}{w_{(p+1),\gamma}} \right)^2 = \sum_{\beta_j \nparallel_\alpha^\gamma \beta_k} \left(\frac{1}{n_{\alpha,k}} \cdot \frac{w_{(p-1),\alpha}^2}{w_{p,k}^2} + \frac{1}{n_{\gamma,k}} \cdot \frac{w_{p,k}^2}{w_{(p+1),\gamma}^2} \right).$$

Let $0 < p < r$. Since the $(p+1)$ -skeleton is pure, we know that every p -cell β_k has at least one transverse neighbour β_j for each $(p-1)$ -cell $\alpha \in \partial\beta_k$. Since there are precisely $n_{\alpha,k}$ transverse neighbours of β_k via α , we have

$$\sum_{\alpha < \beta_k} \left(\frac{w_{(p-1),\alpha}}{w_{p,k}} \right)^2 = \sum_{\beta_j \uparrow_{\alpha}^{\gamma} \beta_k} \frac{1}{n_{\alpha,k}} \cdot \frac{w_{(p-1),\alpha}^2}{w_{p,k}^2}.$$

Similarly, we obtain

$$\sum_{\gamma > \beta_k} \left(\frac{w_{p,k}}{w_{(p+1),\gamma}} \right)^2 = \sum_{\beta_j \uparrow_{\alpha}^{\gamma} \beta_k} \frac{1}{n_{\gamma,k}} \cdot \frac{w_{p,k}^2}{w_{(p+1),\gamma}^2}.$$

That we obtain Forman's decomposition in the special case mentioned follows immediately from Formula 1.4 and Corollary 1.4.7. \square

The reason that we exclude the cases $p = 0$ and $p = r$ in the previous theorem is that we need transverse neighbours to obtain the correct terms on the diagonals. No vertex and no facet has transverse neighbours according to our definition. It does not help to consider transverse neighbourhood via the empty set for example: This way we produce new parallel neighbours that yield non-zero entries in the Bochner-Laplacian, where the Hodge-Laplacian has zero entries. But if we restrict the weights, we get the following result.

Theorem 2.1.2 (combinatorial Weitzenböck's formula for 0-chains).

Consider a weighted CW-complex M where all vertices get the same weight w_0 . Then the following combinatorial Weitzenböck's formula holds:

$$\Delta^0 = \Delta_0^{\nabla}.$$

This combinatorial formula of Weitzenböck type coincides with Forman's decomposition, since all vertices have neighbours of type \parallel^{γ} only. We remark that this formula does not imply that vertices are always Ricci-flat.

Proof. The argument for non-diagonal entries is the same as in the proof of Theorem 2.1.1. For the diagonal entries we have to compare

$$(\Delta_0)_{kk} = \sum_{\gamma > \beta_k} \left(\frac{w_0}{w_{1,\gamma}} \right)^2 \quad \text{and} \quad (\Delta_0^{\nabla})_{kk} = \sum_{\beta_j \parallel^{\gamma} \beta_k} \left(\frac{w_0}{w_{1,\gamma}} \right)^2.$$

This is trivial, since every edge γ incident to β_k gives rise to a unique \parallel^{γ} -neighbour of β_k and vice versa. \square

2.2 COMBINATORIAL GAUSS-BONNET-FORMULA FOR SURFACES

The famous theorem of Gauß and Bonnet for a smooth and closed 2-manifold M states

$$\int_M K \, dM = 2\pi \cdot \chi(M),$$

where K denotes the Gaussian curvature (which in dimension 2 is a different name for the Ricci curvature) and $\chi(M)$ denotes the Euler number of M . Since the Euler number for a sphere equals 2 and for a torus equals 0, we deduce that neither a sphere nor a torus can be endowed with a Riemannian metric that has negative Ricci curvature everywhere. Forman [26, Theorem 7.3] proved that a closed combinatorial manifold of dimension equal to or larger than 2 admits a subdivision such that every edge has negative Ricci curvature, where a standard set of weights is chosen. Hence, a combinatorial analogue of the theorem of Gauß and Bonnet cannot exist in this setting. The situation is different if we consider the geometric set of weights and the (condensed) combinatorial Ricci curvature introduced in Section 1.5 for a *cellular surface*. By a *cellular surface* we mean a pure 2-dimensional CW-complex that is quasiconvex and finite. It may have boundary or not. We are able to prove the following combinatorial version of the theorem of Gauß and Bonnet and remark that in contrast to the classical version no correction is needed for the boundary:

Theorem 2.2.1 (combinatorial Gauß–Bonnet theorem).

Let M be a cellular surface. Consider the geometric set of weights on the cells of M . Then

$$\sum_{k \in [f_1]} (\text{Ric}_1)_{kk} = 4 \cdot \chi(M).$$

Proof. For each parallel neighbour of $\beta_k \in K_1$ there is either a connecting 1-cell α or a connecting 2-cell γ , but not both. For each such α there are $\deg(\alpha) - 3$ many 1-cells β_j with $\beta_j \parallel_\alpha \beta_k$, and for each such γ there are $\text{sides}(\gamma) - 3$ many 1-cells β_j with $\beta_j \parallel^\gamma \beta_k$. To prove the combinatorial Weitzenböck's formula (Theorem 2.1.1) we showed

$$\sum_{\beta_j \parallel_\alpha^\gamma \beta_k} \left(\frac{1}{n_{\alpha,k}} \cdot \frac{w_{(p-1),\alpha}^2}{w_{p,k}^2} + \frac{1}{n_{\gamma,k}} \cdot \frac{w_{p,k}^2}{w_{(p+1),\gamma}^2} \right) = \sum_{\alpha < \beta_k} \left(\frac{w_{(p-1),\alpha}}{w_{p,k}} \right)^2 + \sum_{\gamma > \beta_k} \left(\frac{w_{p,k}}{w_{(p+1),\gamma}} \right)^2.$$

These facts and the choice of weights can be subsumed as follows:

$$\begin{aligned} (\text{Ric}_1)_{kk} &= - \sum_{\beta_j \parallel_\alpha \beta_k} \frac{1}{\deg(\alpha)} - \sum_{\beta_j \parallel^\gamma \beta_k} \frac{1}{\text{sides}(\gamma)} - \sum_{\beta_j \parallel_\alpha^\gamma \beta_k} \left(\frac{1}{n_{\alpha,k}} \cdot \frac{1}{\deg(\alpha)} + \frac{1}{n_{\gamma,k}} \cdot \frac{1}{\text{sides}(\gamma)} \right) \\ &= - \sum_{\alpha < \beta_k} \frac{\deg(\alpha) - 3}{\deg(\alpha)} - \sum_{\gamma > \beta_k} \frac{\text{sides}(\gamma) - 3}{\text{sides}(\gamma)} + \sum_{\alpha < \beta_k} \frac{1}{\deg(\alpha)} + \sum_{\gamma > \beta_k} \frac{1}{\text{sides}(\gamma)} \\ &= 4 \sum_{\alpha < \beta_k} \frac{1}{\deg(\alpha)} + 4 \sum_{\gamma > \beta_k} \frac{1}{\text{sides}(\gamma)} - 4. \end{aligned}$$

Summing over all 1-cells β_k and using the identities

$$f_0 = \sum_{\alpha \in K_0} \deg(\alpha) \cdot \frac{1}{\deg(\alpha)} = \sum_{\alpha \in K_0} \sum_{\beta_k > \alpha} \frac{1}{\deg(\alpha)} = \sum_{k \in [f_1]} \sum_{\alpha < \beta_k} \frac{1}{\deg(\alpha)},$$

and

$$f_2 = \sum_{\gamma \in K_2} \text{sides}(\gamma) \cdot \frac{1}{\text{sides}(\gamma)} = \sum_{\gamma \in K_2} \sum_{\beta_k < \gamma} \frac{1}{\text{sides}(\gamma)} = \sum_{k \in [f_1]} \sum_{\gamma > \beta_k} \frac{1}{\text{sides}(\gamma)},$$

we end up with

$$\begin{aligned} \sum_{k \in [f_1]} (\text{Ric}_1)_{kk} &= \sum_{k \in [f_1]} \left(4 \sum_{\alpha < \beta_k} \frac{1}{\deg(\alpha)} + 4 \sum_{\gamma > \beta_k} \frac{1}{\text{sides}(\gamma)} - 4 \right) \\ &= 4(f_0 - f_1 + f_2) \\ &= 4\chi(M). \end{aligned}$$

□

2.3 A COMBINATORIAL VERSION OF BOCHNER'S THEOREM FOR 1-CHAINS

This interlude describes Forman's proof for a combinatorial version of Bochner's theorem for the first Betti number $b_1 = \dim H_1(M; \mathbb{R})$. For this some additional structure is needed. Instead of quasiconvex CW-complexes, the polyhedral complex is assumed to be a combinatorial manifold (or PL-manifold). This condition assures that the dual complex M^* is well-defined and the associated face lattice is the opposite face lattice of M . We recollect from Stallings [57, Definition 4.4.10] that a combinatorial n -manifold M is a polyhedron such that for each point its link is either an $(n-1)$ -cell or an $(n-1)$ -sphere. We refer to Stallings for the definition of a polyhedron and links of non-simplicial complexes.

Theorem 2.3.1 (“Bochner's theorem for 1-chains”, [26]).

Let M be a compact connected combinatorial n -manifold satisfying $\text{Ric}^F \geq 0$.

1. (Corollary 4.3) Suppose there is a vertex v such that $\text{Ric}(e) > 0$ for all edges e that contain v . Then $b_1 = H_1(M; \mathbb{R}) = 0$.
2. (Theorem 4.4) Suppose $n \leq 3$. Then $b_1 = H_1(M; \mathbb{R}) \leq n$.
3. (Theorem 4.5) Suppose that the dual complex M^* of M contains an n -simplex or an n -cube. Then $b_1 = \dim H_1(M; \mathbb{R}) \leq n$.

By Forman's postulated Weitzenböck's formula mentioned in the introduction, we have

$$\Delta_1 = \Delta^F + \text{Ric}^F.$$

If we know that Δ^F and Ric^F are positive semidefinite, then we know that

$$b_1 = \dim (\text{Ker } \Delta^F \cap \text{Ker } \text{Ric}^F).$$

So the first step is to ensure positive semidefiniteness of these two matrices. By the curvature condition of the theorem, we only have to check Δ^F . Forman calls a matrix A *strongly non-negative* if it is symmetric and satisfies

$$A_{kk} \geq \sum_{j \neq k} |A_{jk}|.$$

A strongly non-negative matrix is positive semidefinite, [26, Theorem 1.3]. Note that Δ^F is strongly non-negative by definition.

In the following, we specialise to a standard set of weights associated to the cells of a combinatorial manifold. We denote by \sim the equivalence relation on the p -cells induced by the parallel neighbourhood relation, by $\mathcal{C}_p(\beta)$ the parallel equivalence class of a p -cell β , and by $\mathcal{N}_p(M)$ the number of parallel equivalence classes. An equivalence class $\mathcal{C}_p(\beta)$ is *flat* if this class is Ricci-flat, that is, if $\text{Ric}^F(\beta_j) = (\text{Ric}_p^F)_{jj} = 0$ for each j with $\beta_j \in \mathcal{C}_p(\beta)$. The number of flat equivalence classes of M is denoted by $\mathcal{N}_p^0(M)$.

Theorem 2.3.2 ([26], Theorem 1.6).

With the above notation and $v = (v_1, \dots, v_{f_p}) \in \text{Ker } \Delta_p^F$, we have:

1. $\dim \text{Ker } \Delta_p^F \leq \mathcal{N}_p(M)$.
2. If $(\Delta_p^F)_{jk} \neq 0$ and $j \neq k$ (i.e. $\beta_j \parallel \beta_k$), then $v_k = -(\text{sign}(\Delta_p^F)_{jk})v_j$.
3. For all $\beta_j \in \mathcal{C}_p(\beta_k)$ the value of v_j is determined by v_k .
4. If $v_k = 0$ for some $k \in f_p(M)$, then $v_j = 0$ for all j with $\beta_j \in \mathcal{C}_p(\beta_k)$.

We remark that this theorem remains true if we replace Δ^F by a strongly non-negative matrix and consider an equivalence relation \sim that is induced by the neighbourhood relation instead of the parallel neighbourhood relation. Forman states Theorem 2.3.2 for strongly non-negative matrices.

Corollary 2.3.3 ([26], Corollary 1.7).

We use the above notation and suppose that Ric_p^F is positive semidefinite.

1. If each equivalence class has a representative β_j with $(\text{Ric}_p^F)_{jj} > 0$, then $\text{Ker } \Delta_p = 0$.
2. The result of Theorem 2.3.2 (1) can be sharpened: $\dim \text{Ker } \Delta_p^F \leq \mathcal{N}_p^0(M)$.

Theorem 2.3.4 ([26], Theorem 2.7 and 2.8).

If Ric_p^F is positive semidefinite, then $b_p = \dim H_p(M; \mathbb{R}) \leq \mathcal{N}_p^0(M) \leq \mathcal{N}_p(M)$.

Corollary 2.3.5 ([26], Corollary 2.9).

We use the above notation and suppose that Ric_p^F is positive definite, that is, each p -cell has positive Ricci curvature. Then $b_p = H_p(M; \mathbb{R}) = 0$.

The keys to Bochner's theorem for 1-chains are Theorem 2.3.7 and Theorem 2.3.8.

Lemma 2.3.6 ([26], Lemma 4.1).

Using the above notation we suppose that $c = \sum_{\beta \in K_1} c_\beta \beta \in \text{Ker } \delta_2 \cap \text{Ker } \Delta_1^F$. Let γ be a 2-face and α be a vertex contained in the boundary of γ . Suppose there are two edges β' and β'' that are transverse neighbours, that is, $\beta' \cap_\alpha^\gamma \beta''$. Moreover, suppose that $c_{\beta'} = c_{\beta''} = 0$. Then

$$c|_{\partial\gamma} = \sum_{\beta \in \partial\gamma} c_\beta \beta = 0.$$

This lemma is needed to prove the next theorem.

Theorem 2.3.7 (“Unique Continuation Theorem”, [26], Theorem 4.2).

Using the above notation and suppose that $c = \sum_{\beta \in K_1} c_\beta \beta \in \text{Ker } \delta_2 \cap \text{Ker } \Delta_1^F$. Suppose, in addition, that there exists a vertex α such that $c_\beta = 0$ for all $\beta > \alpha$. Then $c = 0$.

Forman describes in Section 5 of [26] some problems related to unique continuation theorems for p -chains. We discuss this topic in Section 2.5. To formulate Theorem 2.3.8, we need a few definitions. Consider a *local equivalence class* $\mathcal{C}_\alpha(\beta)$ of an edge β induced by the parallel neighbourhood relation for a vertex α with $\alpha < \beta$: Denote by $K_1(\alpha)$ all 1-cells of M that contain α and define an equivalence relation on $K_1(\alpha)$ induced from the parallel neighbourhood relation by only using edges in $K_1(\alpha)$. The number of such equivalence classes is denoted by \mathcal{N}_α and used to define the *local homology dimension* $D_1(\alpha)$ at α :

$$D_1(\alpha) := \begin{cases} \mathcal{N}_\alpha & |\mathcal{C}_\alpha(\beta)| > 1 \text{ for each equivalence class } \mathcal{C}_\alpha(\beta) \\ \mathcal{N}_\alpha - 1 & \text{there exists an equivalence class } \mathcal{C}_\alpha(\beta) \text{ with } |\mathcal{C}_\alpha(\beta)| = 1. \end{cases}$$

The *homology dimension* D of M is defined by

$$D := \inf_{\alpha \in K_1} D_1(\alpha).$$

This notation is motivated by the following theorem.

Theorem 2.3.8 ([26], Theorem 4.5).

If M has non-negative Ricci curvature Ric^F for every edge, then

$$b_1 = \dim H_1(M; \mathbb{R}) \leq D.$$

The goal is therefore an upper bound for the homology dimension D of M which is difficult in general but easy if the dual complex M^* of M contains an n -simplex or an n -cube. In these cases we have $D \leq n$. For combinatorial manifolds of dimension less than four the condition on the dual complex can be dropped.

2.4 A SECOND EXAMPLE AND BOCHNER'S THEOREM FOR 1-CHAINS

In Section 1.2 we gave an overview over the computations we encountered by studying the boundary of a 3-cube. The aim now is rather to discuss the effect of different choices of weights and decompositions on Bochner's theorem for 1-chains. We do this by a thorough analysis of two different cell decompositions of the 2-dimensional torus with a standard and a geometric set of weights assigned to the cells. As cell decompositions of the torus, we consider the *standard cubical* cell decomposition T_c as shown in Figure 2.1 and *Möbius' torus* T_M as shown in Figure 2.2. We compute the combinatorial Ricci curvature for the standard set of weights with $w_\alpha \equiv w_\beta \equiv w_\gamma \equiv 1$ and for the geometric set of weights for both cell decompositions. Recollect from the proof of Weitzenböck's formula (Theorem 2.1.1) that

$$\sum_{\alpha < \beta_k} \left(\frac{w_{(p-1),\alpha}}{w_{p,k}} \right)^2 + \sum_{\gamma > \beta_k} \left(\frac{w_{p,k}}{w_{(p+1),\gamma}} \right)^2 = \sum_{\beta_j \nabla_\alpha^\gamma \beta_k} \left(\frac{1}{n_{\alpha,k}} \cdot \frac{w_{(p-1),\alpha}^2}{w_{p,k}^2} + \frac{1}{n_{\gamma,k}} \cdot \frac{w_{p,k}^2}{w_{(p+1),\gamma}^2} \right).$$

Together with Corollary 1.5.11 we obtain that the Ricci curvature of an edge β is given by

$$\text{Ric}(\beta) = \#\{\alpha \in \partial\beta\} + \#\{\gamma \in \delta\beta\} - \#\{\beta' \mid \beta' \parallel \beta\}.$$

We now show that the curvature is constant for both sets of weights and both cell decompositions. In case of a standard set of weights, we have for every edge β

$$\text{Ric}_{T_c}(\beta) = 2 + 2 - 4 = 0 \quad \text{and} \quad \text{Ric}_{T_M}(\beta) = 2 + 2 - 6 = -2.$$

Möbius' torus is thus an example that a combinatorial analogue of the theorem of Gauß and Bonnet does not hold if we consider a combinatorial Ricci curvature for a standard set of weights. If we choose the geometric set of weights, we know from Theorem 2.2.1 that $\sum \text{Ric}(\beta) = 0$ is true for *every* quasiconvex cell decomposition of the torus. Recall from the proof of Theorem 2.2.1 that for the geometric set of weights

$$\text{Ric}(\beta) = 4 \sum_{\alpha < \beta} \frac{1}{\deg(\alpha)} + 4 \sum_{\gamma > \beta} \frac{1}{\text{sides}(\gamma)} - 4,$$

(different cell decompositions are already taken into account by this formula), so that we have Ricci-flat complexes T_c and T_M in case of the geometric set of weights:

$$\text{Ric}_{T_c}(\beta) = 4 \cdot \frac{2}{6} + 4 \cdot \frac{2}{3} - 4 = 0 \quad \text{and} \quad \text{Ric}_{T_M}(\beta) = 4 \cdot \frac{2}{6} + 4 \cdot \frac{2}{3} - 4 = 0.$$

We mention that Möbius' torus is *negatively Ricci curved* if we consider the geometric set of weights together with Forman's notion Ric^F .

The combinatorial Gauß–Bonnet–Theorem 2.2.1 obviously guarantees only that the torus is flat *on average*, that is, negatively and positively Ricci-curved edges balance out. A stellar subdivision of an arbitrary triangle of T_M together with the geometric set of weights yields an example for a non Ricci-flat torus.

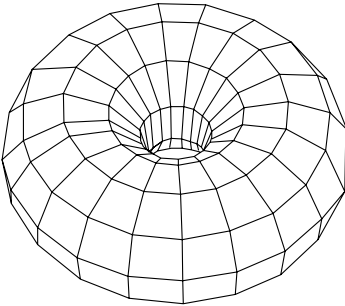


Figure 2.1: The “standard cubical” quasiconvex cell decomposition T_c of the 2-torus is obtained from the standard cubical grid of \mathbb{R}^2 as the quotient of a \mathbb{Z}^2 -action.

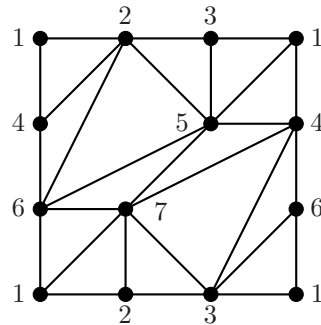


Figure 2.2: Möbius' torus T_M is obtained from tiling \mathbb{R}^2 by the shown triangulation of the square as the quotient of a \mathbb{Z}^2 -action.

As computed, Möbius' torus does not have non-negative Ricci curvature for a given standard set of weights, so Theorem 2.3.1 does not apply. But Möbius' torus is Ricci-flat in the sense of Section 1.5 with geometric set of weights assigned to the cells. In particular, the Bochner-Laplacian Δ^∇ of T_M with the geometric set of weights is positive semidefinite and the dimension of its kernel equals the first Betti number. Nevertheless, Δ^∇ is not strongly non-negative in this case and the upper bound for the first Betti number by the homology dimension as described in the preceeding section does not hold for this “non-standard-weight”-Ricci curvature: The local homology dimension of Möbius' torus at every vertex is 1, an equivalence class contains 6 elements, and the first Betti number of the torus is 2.

We end this section with some general remarks. The condensed Bochner-Laplacian Δ^∇ as described in Section 1.4 is not a strongly non-negative matrix unless the entries $(\Delta^\nabla)_{jk}$ that correspond to transverse neighbours vanish. This happens if we choose a standard set of weights. The first condensed combinatorial Bochner-Laplacian Δ_1^∇ is not even a positive semidefinite matrix in general. But as we have seen for Möbius' torus above, even if the Bochner-Laplacian Δ_p^∇ is positive semidefinite, the method presented in Section 2.3 is not applicable for arbitrary choices of weights.

Thus there is only hope for combinatorial Bochner-type theorems for p -chains if we restrict to the special case that the entries of the Hodge-Laplacian that correspond to transverse neighbours vanish. This guarantees that the condensed Bochner-Laplacian is strongly non-negative. Therefore we shall choose a standard set of weights in Section 2.6.

2.5 PROBLEMS WITH p -CHAINS.

In Section 2.3 we outlined Forman's proof for a combinatorial version of Bochner's theorem for 1-chains. Forman discusses in Section 5 of [26] differences between the smooth Laplacians and their combinatorial counterparts, as well as difficulties one has to cope with if one tries to prove a unique continuation theorem for p -chains. This section is a brief discussion of the problems he discovers for a possible extension. For a smooth Riemannian manifold M , it is known that that p^{th} Betti number b_p is bounded from above by $\binom{n}{p}$ and this bound is sharp if M is a torus. This result can be shown using the *Bochner technique* that uses Weitzenböck's formula for p -chains, see Bérard [10] or Wu [67]. In the combinatorial setting only results for 1-chains are proven so far.

Example 2.5.1 ([26], Example 5.5).

This example points to a problem one faces in one possible generalisation of the Unique Continuation Theorem 2.3.7 for 1-chains to a Unique Continuation Theorem for p -chains. We start with a quasiconvex CW-complex M_1 weighted by a standard set of weights. Let $\omega = \sum_{\beta \in K_p} \lambda_\beta \beta$ be a non-zero p -chain that satisfies $\omega \in \text{Ker } \Delta_p(M_1) \cap \text{Ker } \Delta_p^\nabla(M_1)$. Suppose there is a $(p-1)$ -cell $\alpha \in M_1$ such that $\lambda_\beta = 0$ for every $\beta > \alpha$.

Such M_1 , ω , and α do exist. For example, consider as M_1 a 3-dimensional torus obtained from \mathbb{R}^3 with the standard cubical grid factorised by an appropriate \mathbb{Z}^3 -action. The

canonical generators for $C_2(M_1; \mathbb{R})$ are c^{xy} , c^{xz} , and c^{yz} where c^{xy} has coefficient 1 for the 2-cells parallel to the xy -plane and 0 otherwise. The 2-chain $\omega = c^{xy}$ is now an example for a non-vanishing harmonic chain that has vanishing coefficients for an edge α parallel to the z -direction.

Now consider a disjoint copy M_2 of M_1 where a copied cell $\gamma' \in M_2$ is assigned the weight from its preimage $\gamma \in M_1$. Construct the complex M from M_1 and M_2 by identifying the closure of α with the closure α' and define a p -chain ω' that coincides with ω on M_1 and vanishes on M_2 . Thus we have a non-zero harmonic p -chain that is contained in the kernel of the Bochner-Laplacian of M and vanishes on M_2 .

This problem concerning the unique continuation problem for certain harmonic p -chains relies on the existence of certain M_1 , ω , and α , where ω is a p -chain and α is a $(p-1)$ -cell. As we have seen above, such objects exist. But if we require ω to vanish *around* a *vertex* α instead of *around* a $(p-1)$ -cell then such α and ω do at least not exist on a standard cubical torus. We therefore study a possible unique continuation theorem in the following section where we assume that a 2-chain vanishes *around* a vertex. We restrict to the case of 2-chains in order to focus on problems that occur in this setting and are able to prove a unique continuation theorem under an additional assumption.

For a standard cubical n -dimensional torus T_c^n with a standard set of weights, we can go even further in the analogy to the smooth setting. As in the 2-dimensional case, T_c^n is Ricci-flat with respect to p -cells, that is, the p^{th} condensed combinatorial Ricci curvature vanishes for each p -cell. Hence the p^{th} Betti number can be estimated according to Corollary 2.3.3(2) by the number of (Ricci-flat) parallel equivalence classes which equals $\binom{n}{p}$. At the end of the next section we discuss the relation of the local homological dimension with the Betti numbers for this particular example.

2.6 UNIQUE CONTINUATION THEOREMS FOR 2-CHAINS.

The Example 2.5.1 discussed in the preceeding Section 2.5 helps us to guess a possible version of a unique continuation theorem or a Bochner's theorem for p -chains with $p > 1$. In this section, our aim is to formulate and prove an analogue of Lemma 2.3.6 and of the Unique Continuation Theorem 2.3.7 for 2-chains on quasiconvex combinatorial n -manifolds. Throughout this section we assign a standard set of weights. We recall once more that in this case Forman's Bochner-Laplacian Δ^F coincides with the Bochner-Laplacian Δ^∇ computed in Section 1.4, that is, $\Delta^F = \Delta^\nabla$, and it is strongly non-negative. In particular, it is positive semidefinite and we gain some control over its kernel as described by Theorem 2.3.2.

Lemma 2.6.1. Suppose M is a compact quasiconvex combinatorial n -manifold weighted by a standard set of weights. Let $c = \sum_{j \in [f_2]} c_j \beta_j$ be a 2-chain such that $c \in \text{Ker } \Delta_2^\nabla \cap \text{Ker } \delta_3$ and let γ be a 3-cell. Suppose there are three 2-cells β_r , β_s , and β_t contained in the boundary of γ that have a common vertex v . Suppose that $c_r = c_s = c_t = 0$. Then $c_j = 0$ for $j \in [f_2]$ with $\beta_j \in \partial\gamma$.

Proof. The proof proceeds in two steps: Firstly, we show that $c_k = 0$ if $\beta_k \in \partial\gamma$ and $\beta_k > v$. Secondly, we conclude that $c_k = 0$ for all other k with $\beta_k \in \partial\gamma$.

So let us assume that $k \in [f_2]$ with $\beta_k \in \partial\gamma$, $\beta_k > v$ and $k \notin \{r, s, t\}$. Then β_k is parallel-equivalent to at least one of β_r , β_s , and β_t and the first claim follows by Theorem 2.3.2 (4). That β_k is parallel to at least one of β_r , β_s , and β_t follows from the fact that β_k has two transverse neighbours in $\partial\gamma$ that contain v .

The second step is handled as follows. Assume that $k \in [f_2]$ with $\beta_k \in \partial\gamma$ such that β_k does not contain v . Denote the number of edges in the boundary of β_k by s and let these edges $\alpha_0, \dots, \alpha_s = \alpha_0$ be arranged cyclically, that is, $\alpha_i \cap \alpha_{i+1}$ is a vertex for $0 \leq i < s$. Moreover, assume that β_k is not parallel to any 2-cell β' in the boundary of γ that contains v , otherwise, we know by Theorem 2.3.2 (4) that $c_k = 0$. Therefore $\beta_k \cap \beta'$ is an edge for all $\beta' \in \partial\gamma$ with $\beta' > v$. If t is the number of such 2-faces β' , we have $3 \leq t \leq s$. We denote these 2-faces by β'_ℓ where $\ell \in [t]$ such that $\beta'_\ell \cap \beta_k = \alpha_{j_\ell}$ and $j_\ell < j_{\ell+1}$. In particular, we know that $\beta'_{\ell-1}$ and β'_ℓ intersect in an edge that contains v for each $\ell \in [t]$. If $\beta'_{\ell-1}$, β'_ℓ , and β_k do not intersect in a vertex for some $\ell \in [t]$, then these cells bound a 2-ball. Each 2-cell in such a ball is parallel to each 2-cell β'_m with $m \in [t] \setminus \{\ell-1, \ell\}$. Therefore we have shown that each 2-face β of $\partial\gamma$ has coefficient $c_\beta = 0$ except possibly β_k . Now consider the 2-chain $\partial_3\gamma = \sum_{j \in [f_2]} \lambda_j \beta_j$ with $\lambda_j = [\gamma : \beta_j] \neq 0$ determined by the orientations chosen. In particular, we have $g(\partial_3\gamma, c) = \lambda_k c_k$. But $c \in \text{Ker } \delta_3$ yields

$$\lambda_k c_k = g(\partial_3\gamma, c) = g(\gamma, \delta_3 c) = 0.$$

Hence we have shown that $c_{\beta_k} = 0$ which proves $c_{\partial\gamma} = 0$. \square

The first lemma we need on the way to a combinatorial version of Bochner's theorem for 2-chains is therefore easy to prove. An analogue of the unique continuation theorem is more difficult. It is straightforward for 2-chains on 3-manifolds, but more delicate for manifolds of higher dimension. For this reason we first give a proof of the 3-dimensional case, and discuss some problems that occur in higher dimensions afterwards.

Theorem 2.6.2 (Unique Continuation Theorem for 2-chains on 3-manifolds).

Let M be a connected quasiconvex combinatorial 3-manifold M weighted by a standard set of weights. Suppose that the 2-chain $c = \sum_{j \in [f_2]} c_j \beta_j$ satisfies $c \in \text{Ker } \Delta_2^\nabla \cap \text{Ker } \delta_3 \cap \text{Ker } \partial_2$. Suppose in addition that there is a vertex v such that $c_k = 0$ for all 2-cells β_k that contain v . Then $c = 0$.

Proof. A (finite) edge path between two vertices v_1 and v_2 is a collection of edges e_1, \dots, e_ℓ such that e_i and e_{i+1} have a common vertex and v_1 (resp. v_2) is the endpoint of e_1 (resp. e_ℓ) that is not the common vertex with e_2 (resp. $e_{\ell-1}$). The length of an edge path is the number of edges used. Define a distance function D_v on the vertices of M via

$$D_v(v') := \min \{ |\text{length}(\tau)| \mid \tau \text{ is an edge path between } v \text{ and } v' \}.$$

We prove the theorem inductively. From the hypothesis of the theorem and Lemma 2.6.1 we know that $c_j = 0$ for every 2-face β_j that is contained in the boundary of a 3-face that

contains v . We now assume that every vertex w of distance less than or equal to k from v has this property, that is, we have $c_{|\partial\gamma} = 0$ for each 3-face γ that contains w . We show that a vertex v' of distance $k + 1$ from v has this property, too.

Consider the edge $\alpha = \{v', w\}$ that connects the vertices v' with $D_v(v') = k + 1$ and w of distance $D_v(w) = k$. Define

$$\begin{aligned}\Gamma_\alpha &:= \{\gamma \in K_3 \mid \alpha \text{ is a face of } \gamma\}, \\ E_{v'}(\alpha) &:= \{\alpha' \in K_1 \mid \alpha' \neq \alpha \text{ and there is } \gamma \in \Gamma_\alpha \text{ such that } v' < \alpha' < \gamma\},\end{aligned}$$

and partition $E_{v'}(\alpha)$ as follows:

$$\begin{aligned}A_{v'}(\alpha) &:= \{\alpha' \in E_{v'}(\alpha) \mid \text{there are } \gamma \in \Gamma_\alpha \text{ and } \beta \in \partial\gamma \text{ such that } \alpha, \alpha' \in \partial\beta\}, \\ B_{v'}(\alpha) &:= E_{v'}(\alpha) \setminus A_{v'}.\end{aligned}$$

A partial view of the boundary of a 3-cell in Γ_α is shown in Figure 2.3. Examples for edges in $A_{v'}(\alpha)$ and $B_{v'}(\alpha)$ are also given there. Every 2-face β that is contained in the boundary $\partial\gamma$ for some $\gamma \in \Gamma_\alpha$ is not contained in the support of c by Lemma 2.6.1. We now prove that every 2-face β' that contains an edge $\alpha' \in A_{v'}(\alpha)$ or β'' that contains an edge $\alpha'' \in B_{v'}(\alpha)$ is not contained in the support of c , too.

Case 1: $\alpha' \in A_{v'}(\alpha)$. A picture of this case is given by Figure 2.4. Let $\gamma \in \Gamma_\alpha$ be a 3-face with $\alpha' < \gamma$ and $\beta_0 \in \partial\gamma$ a 2-face such that α and α' are contained in $\partial\beta_0$. In particular, $c_{\beta_0} = 0$. Let β' be a 2-face that contains α' . Either β' is a transverse neighbour of β_0 or not. If β' is a transverse neighbour of β_0 , then it is contained in a 3-face that contains w . Hence $c_{\beta'} = 0$ by hypothesis. If β' is a parallel neighbour of β_0 , then $c_{\beta'} = 0$ by Theorem 2.3.2.

Case 2: $\alpha'' \in B_{v'}(\alpha)$. Since α'' is contained in at least three 3-faces, we distinguish two cases. Either α'' is contained in precisely three 2-faces (and therefore three 3-faces) or it is contained in more than three 2-faces. Illustrations of these cases can be found in Figures 2.5 and 2.6. We denote the 3-face that contains α and α'' by γ and the two 2-faces on the boundary of γ that contain α'' by β and β'' . If β'' is the only 2-face that contains α''

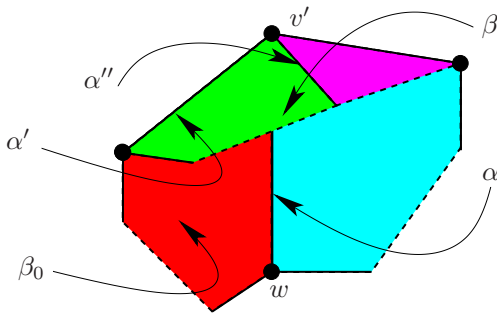


Figure 2.3: A partial view of the boundary $\partial\gamma$ of a 3-face γ . Dashed lines symbolise the boundary of a 2-face. $\alpha' \in A_{v'}(\alpha)$ and $\alpha'' \in B_{v'}(\alpha)$.

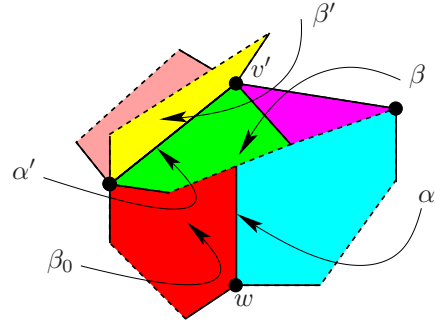


Figure 2.4: (Case 1) The 2-faces β_0 and β' are either contained on the boundary of one 3-face or not. If not they are parallel neighbours.

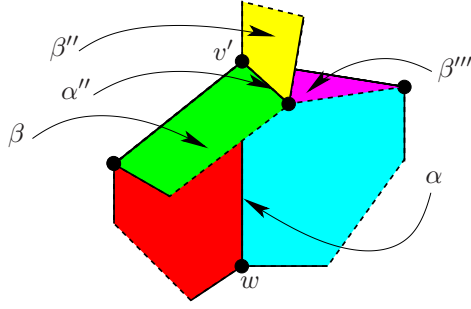


Figure 2.5: To conclude that $c_{\beta''} = 0$ we use that $c_{\beta} = c_{\beta'''} = 0$ (they are on the boundary of 3-faces that contain w) and that $c \in \text{Ker } \partial$.

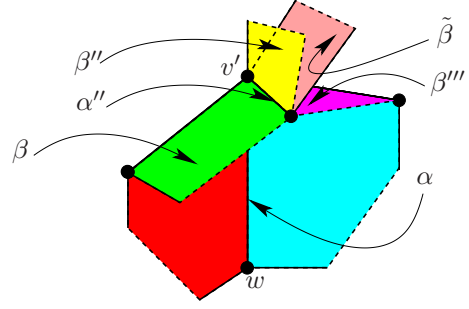


Figure 2.6: To conclude that $c_{\beta''} = 0$ we use that β' is a parallel neighbour of β or β''' and that $c_{\beta} = c_{\beta'''} = 0$.

and is *not* contained in $\partial\gamma$, then we have, since $c \in \text{Ker } \partial_2$

$$0 = g(\partial_2 c, \alpha'') = g(c, \delta_2 \alpha'') = g(0 \cdot \beta + 0 \cdot \beta''' + c_{\beta''} \beta'', \lambda_{\beta} \beta + \lambda_{\beta''} \beta'' + \lambda_{\beta'''} \beta''') = c_{\beta''} \lambda_{\beta''},$$

where λ_{β} , $\lambda_{\beta''}$, and $\lambda_{\beta'''}$ are non-zero constants depending on the chosen orientation. If there are more than three faces of dimension 2 that contain α'' , then each 2-face β'' that is not contained in the boundary of γ is a parallel neighbour of either β , β''' , or both. It follows $c_{\beta''} = 0$.

Hence, we have shown that $c_{\beta} = 0$ for each β that contains an edge in $E_{v'}(\alpha)$. In particular, we have $c|_{\partial\gamma'} = 0$ for all 3-cells γ' that contain at least two edges from $E_{v'}(\alpha)$ by Lemma 2.6.1. What about a 3-face γ'' that contains only one such edge? An example is given in Figure 2.7 where β'' , $\tilde{\beta}$, and β^{\square} are on the boundary of such a γ'' . So far we know that the two 2-faces β'' , $\tilde{\beta} \in \partial\gamma''$ that contain α'' in their boundary are not contained in the support of c , that is, $c_{\beta''} = c_{\tilde{\beta}} = 0$. To apply Lemma 2.6.1, we need one more 2-face of $\partial\gamma''$ that is not contained in the support of c but that contains v . To identify such a third 2-face, we consider the 3-face γ that contains the edges α and α'' . As shown in Figure 2.7, we denote the 2-faces on the boundary of γ that contain α'' by β and β''' .

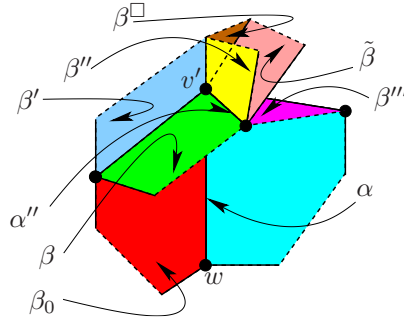


Figure 2.7: To conclude that $c_{\beta^{\square}} = 0$ we use that the 3-cells “around” α can be ordered in a cyclic way. Here $c_{\beta'} = 0$ implies $c_{\beta^{\square}} = 0$.

The s 3-faces “around” α'' can be ordered in a cyclic way to form a non-trivial sequence $\gamma_0 = \gamma, \gamma_1, \dots, \gamma_s = \gamma$ such that $\partial\gamma_j \cap \partial\gamma_{j+1}$ is a 2-face that contains α'' for $0 \leq j < s$. For example, we can arrange everything such that $\partial\gamma_0 \cap \partial\gamma_1 = \beta$ and $\partial\gamma_{s-1} \cap \partial\gamma_s = \beta'''$. As in Case 2 above, we can show that $c|_{\partial\gamma_j} = 0$ for each $2 \leq j \leq s-2$, while we already know that $c|_{\partial\gamma_1} = c|_{\partial\gamma_{s-1}} = 0$ since both 3-faces contain two edges of $E_{v'}(\alpha)$. Consider the 2-face β^* of $\partial\gamma_2$ that is different from $\partial\gamma_1 \cap \partial\gamma_2$ and that contains the edge $\partial\beta' \cap \partial\beta''$ (if $\gamma'' = \gamma_2$ we have $\beta^* = \beta^\square$). As in Case 2, we obtain that $c_{\beta^*} = 0$, since either these three 2-faces are the only 2-faces that contain $\partial\beta' \cap \partial\beta''$ or β^* is parallel to one of β' and β'' . Hence, $c|_{\partial\gamma_2} = 0$. This argument can now be iterated to show that $c|_{\partial\gamma_2} = \dots = c|_{\partial\gamma_{s-2}} = 0$.

We now consider a “new layer” of 3-faces Γ_α^1 and edges $E_{v'}^1$, emanating from v' . The idea is to use the 3-faces that are not contained in Γ_α but contain an edge of $E_{v'}$. We have just shown that the boundaries of these 3-faces are not contained in the support of c . More precisely, we consider

$$\begin{aligned}\Gamma_\alpha^1 &:= \{\gamma \in K_3 \setminus \Gamma_\alpha \mid \gamma \text{ contains an edge of } E_{v'}(\alpha)\}, \\ E_{v'}^1(\alpha) &:= \{\alpha' \in K_1 \setminus E_{v'}(\alpha) \mid \text{there is a } \gamma \in \Gamma_\alpha^1 \text{ such that } v' < \alpha' < \gamma\}.\end{aligned}$$

We partition the edges of $E_{v'}^1(\alpha)$ into two sets $A_{v'}^1(\alpha)$ and $B_{v'}^1(\alpha)$, where $A_{v'}^1(\alpha)$ consists of all edges that define a 2-face together with an edge of $E_{v'}(\alpha)$. We now proceed as in case of Γ_α and $E_{v'}(\alpha)$ described above. After a finite number of such layers we have shown of all 3-faces γ that contain v' that $c|_{\partial\gamma} = 0$. \square

The following corollary is a strengthening of Corollary 2.3.5 where *positive* combinatorial Ricci curvature is assumed to obtain the same conclusion.

Corollary 2.6.3 (analogue of [26, Corollary 4.3] for 2nd Betti number).

Suppose M is compact, connected, quasiconvex combinatorial 3-manifold weighted by a standard set of weights and its second combinatorial Ricci curvature is non-negative for every 2-face, i.e., $\text{Ric}(\sigma) \geq 0$ for every 2-cell σ . Suppose there exists a vertex v such that all 2-cells σ that contain v are positively curved, i.e., $\text{Ric}(\sigma) > 0$. Then $H_2(M; \mathbb{R}) = 0$.

Proof. Suppose $c = \sum_{\beta \in K_2} c_\beta \beta \in C_2(M; \mathbb{R})$ satisfies $c \in \text{Ker } \Delta_2$. By the combinatorial formula of Weitzenböck type for 2-chains and since Δ_2^∇ and Ric are positive semidefinite, we have $c \in \text{Ker } \Delta_2^\nabla \cap \text{Ker } \text{Ric}$. Since $\text{Ric}(\beta) > 0$ for every 2-face β that contains v and since $c \in \text{Ker } \text{Ric}$ we conclude

$$c_\beta = 0 \quad \text{if } v < \beta.$$

Moreover, if $c \in \Delta_2$ we learn

$$c \in \text{Ker } \Delta_2 \cap \text{Ker } \Delta_2^\nabla = \text{Ker } \partial_2 \cap \delta_3 \cap \text{Ker } \Delta_2^\nabla.$$

Together with Theorem 2.6.2 we conclude $c = 0$. But $H_2(M; \mathbb{R}) \cong \text{Ker } \Delta_2$ implies the corollary. \square

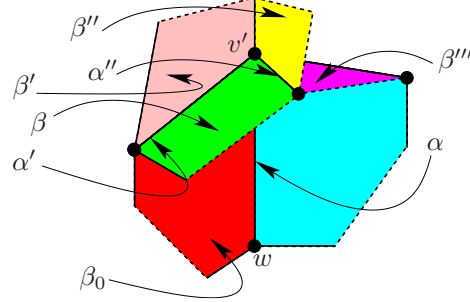


Figure 2.8: We may assume that $\beta''' \in B_{v'}(\alpha)$ and that β''' and β'' intersect in an edge.

Problems arise if we try to extend the unique continuation theorem for 2-chains to an n -manifold for $n > 3$. More precisely, we are not able to imitate Case 2 of the above proof in higher dimensions. In three dimensions, it was easy since a 2-face β' that contains an edge of $B_{v'}(\alpha)$ is either the only transverse neighbour of β and β''' or it is parallel to at least one of them, see Figures 2.5 and 2.6. This is not necessarily true in higher dimensions. We show by example that it does not suffice to search for a parallel-equivalent neighbour with vanishing coefficient in a naïve way. We start with a description of the situation depicted in Figure 2.8. Consider $\alpha = \{v', w\}$, $E_{v'}(\alpha) = A_{v'}(\alpha) \sqcup B_{v'}(\alpha)$ as in the proof of Theorem 2.6.2, $\alpha' \in A_{v'}(\alpha)$, and β_0 the unique 2-face that contains α and α' . Let γ be a 3-face that contains β_0 . Consider the 2-face β of $\partial\gamma$ that is different from β_0 and contains α' and the edge $\alpha'' \in \partial\beta$ that contains v' and is different from α' . Obviously, $\alpha'' \in E_{v'}(\alpha)$. Let γ' be a 3-face that intersects γ in β . Denote the 2-face different from β in the boundary of γ' that contains α' (resp. α'') by β' (resp. β''). From $\alpha' \in A_{v'}(\alpha)$ we deduce as in Case 1 that β' is not in the support of c , that is, $c_{\beta'} = 0$. Similarly if $\alpha'' \in A_{v'}(\alpha)$, so we have $c_{|\partial\gamma'} = 0$ by Lemma 2.6.1. Therefore, we assume that $\alpha'' \in B_{v'}(\alpha)$. Moreover, we can assume without loss of generality that β' and β'' intersect in an edge, otherwise they are parallel neighbours which implies that $c_{\beta''} = 0$ and hence $c_{|\partial\gamma'} = 0$ by Lemma 2.6.1.

The problem is to show that $\beta'' \in \partial\gamma'$ is not in the support of c . The naive way is to show that a 2-face parallel-equivalent to β'' is not in the support of c , where this parallel-equivalent 2-face is a face of γ' . We now give simple examples for γ' that show that this approach does not work. All we can do is exploit the fact $c \in \text{Ker } \delta$ to unveil some relationship between the unknown coefficients of 2-faces of γ' . The assumption $c \in \text{Ker } \delta$ that is used for Case 2 in the proof of Theorem 2.6.2 only yields relations between unknown coefficients of 2-faces that contain α'' in their boundary. So far this can not be analysed. The examples are depicted in Figures 2.9–2.11. The upper 2-faces are translucent, the lower 2-faces are coloured according to their corresponding faces in Figure 2.8, and the upper edges are dashed.

The first example, Figure 2.9, is a simplex. The triangular 2-faces $\beta'' = \{A, B, v'\}$ and $\beta = \{A, B, C\}$ are transverse neighbours. From $c \in \text{Ker } \delta_3$ we deduce that their

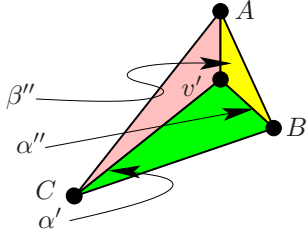


Figure 2.9: A simplex.

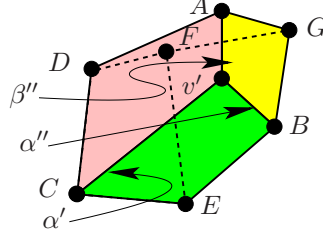


Figure 2.10: A cube.

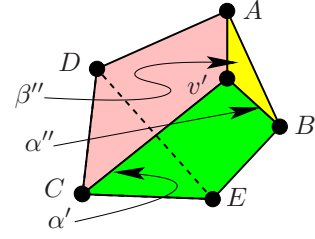


Figure 2.11: A prism over a triangle.

coefficients equal up to sign and that the sign is determined by the orientations chosen:

$$0 = g(\delta_3 c, \gamma') = g(c, \partial_3 \gamma') = c_{\beta''}[\gamma' : \beta''] + c_{\tilde{\beta}}[\gamma' : \tilde{\beta}].$$

Theorem 2.3.2 (2) does not give additional relations since β'' and $\tilde{\beta}$ are transverse neighbours.

If we consider the next example, a cube as depicted in Figure 2.10, we deduce that the coefficients of the parallel 2-faces $\{A, B, v', G\}$ and $\{C, D, E, F\}$ equal up to sign. This time, the sign is determined by Theorem 2.3.2. The other two unknown coefficients for $\{A, D, F, G\}$ and $\{B, E, F, G\}$ are forced to vanish since their corresponding 2-faces are parallel neighbours of 2-faces with vanishing coefficient. Now the technique of the previous example can be applied to obtain additional relations. But the relation obtained this way coincides with the relation obtained from Theorem 2.3.2.

A similar situation holds in example 2.11 for the parallel 2-faces $\{A, B, v'\}$ and $\{C, D, E\}$. This time the coefficient of $\{A, B, D, E\}$ is forced to vanish by the coboundary condition $c \in \text{Ker } \delta$.

To summarise these examples, we have difficulties to show that 2-faces that contain an edge of $B_{v'}(\alpha)$ have a coefficient that vanishes. But if we add one condition, we are able to derive a unique continuation theorem for certain 2-chains on a quasiconvex combinatorial n -manifold. A 2-chain $c \in \text{Ker } \Delta_2^\nabla \cap \Delta_2$ on a quasiconvex combinatorial n -manifold satisfies the *edge-coboundary condition* if for every edge α and every triple β_1, β_2 , and β_3 of pairwise transverse neighbours “via α ” with $c_{\beta_j} = c_{\beta_k} = 0$ we have $c_{\beta_\ell} = 0$. This condition is strong enough to avoid the problems discussed above and implies that the coefficient $c_{\beta''}$ of β'' vanishes as desired.

On a 3-manifold, each $c \in \text{Ker } \Delta_2^\nabla \cap \Delta_2$ satisfies obviously the edge-coboundary condition.

Theorem 2.6.4 (Unique Continuation Theorem for 2-chains on n -manifolds).

Let M be a connected quasiconvex combinatorial n -manifold M weighted by a standard set of weights. Suppose that the 2-chain $c = \sum_{j \in [f_2]} c_j \beta_j$ satisfies $c \in \text{Ker } \Delta_2^\nabla \cap \text{Ker } \delta_3 \cap \text{Ker } \partial_2$ as well as the edge-coboundary condition. Suppose in addition that there is a vertex v such that $c_k = 0$ for all 2-cells β_k that contain v . Then $c = 0$.

The edge-coboundary condition can be used to generalise the proof of Theorem 2.6.2 to 2-chains on n -manifold. As in the proof of Theorem 2.6.2, we consider a vertex v' and

assume that for each vertex w with $D_v(w) < D_v(v')$ the boundary of each 3-face that contains w is not in the support of c . The edge-coboundary condition now implies that each 3-face γ that contains v' in its boundary has three 2-faces that contain v' and that are not in the support of c . By Lemma 2.6.1 we conclude that $\partial\gamma$ is not in the support of c .

It is not clear whether the edge-coboundary condition is too strong to prove Bochner's theorem for 2-chains or not. According to Forman's programme, the next step is to prove that an analogue of the homological dimension is an upper bound for the second Betti number of a *non-negatively curved* quasiconvex n -manifold. In principle, this means to copy the proof of Theorem 4.5 of [26]. Unfortunately, this is not exactly what we have to do, since we added the edge-coboundary condition to the assumptions of the unique continuation theorem for 2-chains. So we prove only an upper bound for the dimension of the harmonic 2-chains that satisfy the edge-coboundary condition. A priori this number is less than or equal to the second Betti number.

We end this section with some observations and remarks. In Section 2.5 we already considered the standard cubical n -dimensional torus T_c^n that is obtained from \mathbb{R}^n as quotient of an appropriate \mathbb{Z}_n -action and a cell decomposition that comes from the grid defined by the coordinate axes. We have seen there that the p^{th} Betti number $b_p(T_c^n)$ equals $\binom{n}{p}$ and that $b_p(T_c^n)$ is given by the number of *global* parallel equivalence classes of p -cells. In case of $p = 2$ we observe that each 2-chain $c \in \text{Ker } \Delta_2^\nabla \cap \text{Ker } \delta_3 \cap \text{Ker } \partial_2$ that vanishes on all 2-cells around a vertex v satisfies the edge-coboundary condition trivially, since each 2-cell β is parallel equivalent to a 2-cell that contains v . Therefore β is not contained in the support of c by Theorem 2.3.2. Let us define the (second) local homological dimension at a vertex v of T_c^n . Forman defined the (first) local homological dimension as the number of equivalence classes induced by the parallel neighbourhood relation on the 1-cells that contain v if each equivalence class contains at least two elements. We define the (second) homological dimension as the number of equivalence classes induced by the parallel neighbourhood relation on the 2-cells that contain v if each equivalence class contains at least two elements. It is easy to see that each 2-face at v has four parallel equivalent neighbours that contain v and there are $\binom{n}{2}$ equivalence classes. Moreover the local homological dimension at v is independent of v . Hence the infimum of the local homological dimensions equals $\binom{n}{2}$. Moreover, the same argument as in Forman's proof of Theorem 2.3.8 in case of 1-chains extends to 2-chains and shows that the (second) local homological dimension is an upper bound for the second Betti number $b_2(T_c^n)$.

But is it possible to extend Theorem 2.3.1? Let us assume that the dimension of the space of 2-chains in $\text{Ker } \Delta_2^\nabla \cap \text{Ker } \delta_3 \cap \text{Ker } \partial_2$ that vanish on all 2-cells around some vertex v and satisfy the edge-coboundary condition equals the dimension of $H_2(M; \mathbb{R})$. Under this assumption we copy Theorem 4.5 of [26] and replace $H_1(M; \mathbb{R})$ by $H_2(M; \mathbb{R})$. We obtain an upper bound of the second Betti number if we additionally assume that none of the local equivalence classes $N_2(v)$ at v contains only one element: $b_2(M) \leq D_2(v)$ where $D_2(v)$ denotes the number of equivalence classes of $N_2(v)$. Let us assume that the dual v^* of v in the dual complex M^* of M is an n -cube. Then each 2-cell that contains v has precisely four (locally) parallel equivalent neighbours that contain v and there are $\binom{n}{2}$ different (local)

equivalence classes. Hence, $b_2(M) \leq \binom{n}{2}$.

Let us still assume that the dimension of the space of 2-chains in $\text{Ker } \Delta_2^\nabla \cap \text{Ker } \delta_3 \cap \text{Ker } \partial_2$ that vanish on all 2-cells around some vertex v and satisfy the edge-coboundary condition equals the dimension of $H_2(M; \mathbb{R})$. We now extend Forman's definition of the (first) local homological dimension $D_1(v)$ to $D_2(v)$ in case that the dual cell v^* of v is a simplex. This implies in particular that each 2-cell that contains v does not have a locally parallel equivalent neighbour that contains v and that there are $\binom{n+1}{n-1} = \binom{n+1}{2}$ many 2-cells that contain v . We define

$$D_2(v) := \binom{n+1}{2} - \binom{n}{1} = \binom{n}{2}.$$

This can be seen as an extension of Forman's definition if one equivalence class contains only one element :

$$D_1(v) := \binom{n+1}{1} - \binom{n}{0} = (n+1) - 1 = n.$$

Moreover, assume that $\binom{n}{2}$ of the 2-cells that contain v are not contained in the support of $c \in H_2(M; \mathbb{R})$. We believe that the remaining $\binom{n}{1} = n$ cells of dimension 2 that contain v are forced by the edge-coboundary condition to be not contained in the support of c . Together with the Unique Continuation Theorem 2.6.4, this would imply that $b_2(M) \leq \binom{n}{2}$ if the dual M^* of M contains a simplex.

2.7 DIAMETER ESTIMATES FOR SOME SIMPLE MANIFOLDS.

The *Hirsch conjecture* is a long-standing problem in linear programming and optimisation that was posed by Warren M. Hirsch in 1957 and reported by Dantzig in 1963 [21]. Hirsch asked for an upper bound on the diameter of the graph of a convex d -polytope with n facets. The conjecture is that the diameter is at most $n - d$, that is, that any two vertices can be joined by an edge-path which consists of at most $n - d$ edges. The number of iterations needed for the simplex algorithm with any pivot rule has certainly the diameter as a lower bound. The quest for a polynomial bound on the diameter (polynomial in n and d) is therefore closely linked to the question whether there is a pivot rule that makes the simplex algorithm strongly polynomial; see the survey by Klee and Kleinschmidt [37]. Only partial answers to Hirsch' question have been given during the last 45 years:

- The Hirsch conjecture is known to be true for $d \leq 3$ and all n , Klee [35],
- For $n - d \leq 5$, Klee and Walkup [38] verified the conjecture.
- The conjecture is sharp for d -cubes.
- Duals of cyclic polytopes satisfy the Hirsch conjecture [36].
- Kalai [34] used the hard Lefschetz theorem to prove a polynomial bound for the diameter of a polytope P that is a dual of a neighbourly polytope:

$$\text{diam } P \leq d^2(n - d)^2 \log n.$$

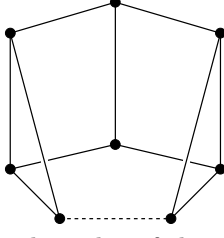


Figure 2.12: The polar of the cyclic polytope on six vertices. The dashed edge is Ricci-flat if a standard set of weights is assigned to the cells.

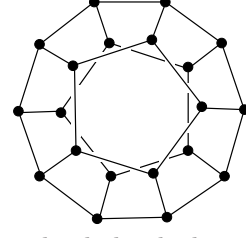


Figure 2.13: The dodecahedron. Every edge is Ricci-flat if we weight the dodecahedron with a standard set of weights.

It is easy to show that it suffices to prove the conjecture for simple convex polytopes, Ziegler [68]. This is one reason why we shall restrict our focus on simple manifolds that are defined later. Another is that some technical difficulties can be avoided this way. If one is interested in non-simple objects, these technicalities can be added later.

Surprisingly, the Hirsch conjecture for all dimensions follows from a special case, the *d-step conjecture*, Klee and Walkup [38]: It suffices to prove for $d \geq 4$ the Hirsch conjecture for simple d -polytopes that have $2d$ facets.

For the Hirsch conjecture as stated, it is important to consider convex polytopes. The Hirsch conjecture is false if we omit this condition, as Mani and Walkup [48] and Barnette [9] have shown. We therefore cannot expect a proof of the Hirsch conjecture if we drop the convexity assumption, but the general theory still may yield interesting upper bounds for the diameter.

The following presentation is an adaptation for simple manifolds of Section 6 of Forman [26] where a combinatorial version of Myers' theorem is proved. His results and proofs are closely related to Myers' original paper [53] and another combinatorial version of Myers' theorem presented by Stone [61, 62]. A crucial condition in Myers' theorem is the assumption that the manifold has positive Ricci curvature everywhere. Forman assumes that the combinatorial Ricci curvature Ric^F with respect to a standard set of weights is positive for each edge to prove a diameter estimate and a combinatorial version of Myers' theorem. Unfortunately, this restriction to a standard set of weights rules out a number of possible candidates. Two elementary examples depicted in Figure 2.12 and Figure 2.13. The dodecahedron is a simple convex polytope that is Ricci-flat for any standard set of weights and the depicted d -step polytope in dimension three has an edge with vanishing Ricci curvature for any standard set of weights.

Our aim in this section is therefore to explore the possibilities to extend Forman's approach to more general choices of weights.

Simple manifolds: As explained above, it is natural to consider *simple objects*. A *simple d -dimensional manifold* M is a closed d -dimensional quasiconvex combinatorial manifold that is the dual of a closed simplicial d -dimensional manifold.

Before we dwell on advantages of simple manifolds, we mention the important but rather trivial fact that every vertex is contained in $d + 1$ edges and any two edges that intersect in a vertex define a 2-face, that is, every edge is contained in d 2-faces.

What makes a simple manifold M particularly nice? Since its dual manifold is simplicial, no edge has a parallel neighbour of type \parallel_α . As an immediate consequence, the combinatorial Ricci curvature of an edge β is

$$\begin{aligned} \text{Ric}(\beta) &= \sum_{\alpha < \beta} \frac{w_{0,\alpha}^2}{w_{1,\beta}^2} + \sum_{\gamma > \beta} \frac{w_{1,\beta}^2}{w_{2,\gamma}^2} - \sum_{\beta_j \parallel_\alpha \beta} \frac{w_{0,\alpha}^2}{w_{1,\beta}^2} - \sum_{\beta_j \parallel_\gamma \beta} \frac{w_{1,\beta}^2}{w_{2,\gamma}^2} \\ &= \sum_{\alpha < \beta} \frac{w_{0,\alpha}^2}{w_{1,\beta}^2} + \sum_{\gamma > \beta} (4 - \text{sides}(\gamma)) \frac{w_{1,\beta}^2}{w_{2,\gamma}^2}. \end{aligned}$$

We see later in Lemma 2.7.1 that simple combinatorial manifolds have another nice property: A *combinatorial Jacobi field* along an arbitrarily given path exists and is uniquely determined by the value at one edge.

Diameters: An *edge-path* ρ between two vertices α and α' is a sequence of vertices and edges $\alpha_0 := \alpha, \beta_1, \alpha_1, \beta_2, \dots, \beta_k, \alpha_k := \alpha'$ such that α_{s-1} and α_s are endpoints of β_s for every s . The *length* of ρ is k . An edge-path is *minimal* if k is smallest possible. This gives rise to the distance between any two vertices:

$$\text{dist}(\alpha, \alpha') := \text{length of a minimal path between } \alpha \text{ and } \alpha'.$$

The *diameter* of a simple d -manifold is the maximum of the distances of any two vertices. Assume there is a positive lower bound c for the Ricci curvature of every edge β . For an edge-path $\rho = \alpha_0, \beta_1, \alpha_1, \beta_2, \dots, \beta_k, \alpha_k$ of length k we obtain the inequality

$$0 < ck \leq \sum_{j=1}^k \text{Ric}(\beta_j) = \sum_{j=1}^k \frac{w_{0,\alpha_{j-1}}^2}{w_{1,\beta_j}^2} + \sum_{j=1}^k \frac{w_{0,\alpha_j}^2}{w_{1,\beta_j}^2} + \sum_{j=1}^k \sum_{\gamma > \beta_j} [4 - \text{sides}(\gamma)] \frac{w_{1,\beta_j}^2}{w_{2,\gamma}^2} \quad (2.1)$$

Jacobi fields (Stone [61] and Forman [26]): Let M be a finite and closed combinatorial d -manifold and $\rho = \alpha_0, \beta_1, \alpha_1, \beta_2, \dots, \beta_k, \alpha_k$ be an edge-path. A (*combinatorial*) *Jacobi field* J along ρ is a map $J : \{\beta_s\}_{1 \leq s \leq k} \rightarrow K_2(M)$ such that the following conditions are satisfied:

1. For all $1 \leq s \leq k$ we have $J(\beta_s) > \beta_s$.
2. For all $1 \leq s \leq k-1$ we have either $J(\beta_s) = J(\beta_{s+1})$ or $J(\beta_s)$ and $J(\beta_{s+1})$ share a 1-cell different from β_s and β_{s+1} .

An example of a Jacobi field along a path ρ is given in Figure 2.14. It is not true that every path admits a Jacobi field, see Figure 2.15. But if a path ρ admits two Jacobi fields J_1 and J_2 that coincide at one edge, then $J_1 = J_2$, Forman [26, Lemma 6.5]. In case of simple manifolds we make the following easy but useful observation.

Lemma 2.7.1. Let M be a simple d -manifold and ρ be any path on M . Choose an edge β of ρ and a 2-face γ with $\gamma > \beta$. Then there is a unique Jacobi field J along ρ with $J(\beta) = \gamma$.

Proof. Here the key property is that parallel neighbours of type \parallel_α do not exist, since the manifold is simple. We want to extend $J(\beta_1)$ from $\alpha_0, \beta_1, \alpha_1$ to $\alpha_0, \beta_1, \alpha_1, \beta_2, \alpha_2$. If β_2 is

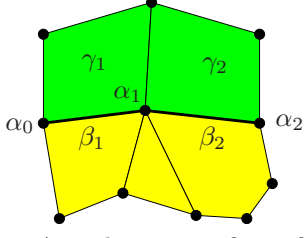


Figure 2.14: A path $\rho = \alpha_0, \beta_1\alpha_1, \beta_2, \alpha_2$ on a cellular surface. There is a Jacobi field J along ρ with $J(\beta_1) = \gamma_1$, and $J(\beta_2) = \gamma_2$. This is the only Jacobi field along ρ .

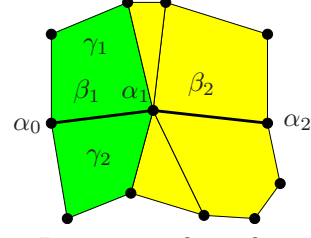


Figure 2.15: Let $\rho = \alpha_0, \beta_1\alpha_1, \beta_2, \alpha_2$ be a path on a cellular surface. There is no Jacobi field along ρ since no 2-face that contains β_2 intersects γ_1 or γ_2 .

contained in the boundary of $J(\beta_1)$, we are done: $J(\beta_2) := J(\beta_1)$. So let us assume that β_2 is not contained in the boundary of $J(\beta_1)$. Let β_3 be the edge in the boundary of $J(\beta_1)$ that contains α_1 and that is different from β_1 . Now β_2 and β_3 have to be transverse neighbours, that is, there is a 2-face γ that contains both edges on its boundary. The Jacobi field extends by $J(\beta_2) := \gamma$. The uniqueness part follows from Forman [26, Lemma 6.5]. \square

The fact that we can extend the Jacobi field from $\alpha_0, \beta_1, \alpha_1$ to $\alpha_0, \beta_1, \alpha_1, \beta_2, \alpha_2$ is denoted by Forman [26] as $NC(\beta_1, \beta_2) = \emptyset$. A careful book-keeping of these entities makes statements for non-simple manifolds possible.

Variations of edge-paths: Any Jacobi field J along a path ρ from α to α' gives rise to a different path ρ_J between these two points as follows: We break ρ into different subpaths with the property that J is constant along each subpath but J differs on consecutive subpaths. For every subpath we now have an alternative path along the 1-cells in the boundary of the associated 2-cell which are not used. Concatenating these alternative paths and deleting the 1-cells travelled successively in opposite directions. We end up with the new path ρ_J from α to α' induced by J , see Figures 2.16 and 2.17.

Let J be a Jacobi field along a path ρ from α to α' that we assume to be minimal. Let us denote the number of maximal J -constant subpaths of ρ denote by $r(J)$. These subpaths are then $\rho_1, \dots, \rho_{r(J)}$. The image of any 1-cell of ρ_s under J is $J(\rho_s)$. Since $\text{length}(\rho_s) \geq 1$

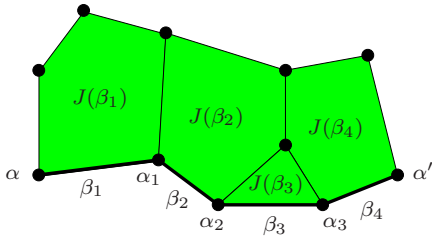


Figure 2.16: A Jacobi field J along the path $\rho = \alpha, \beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3, \beta_4, \alpha'$ from α to α' .

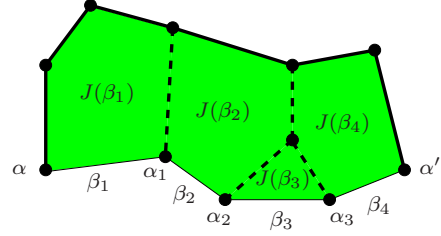


Figure 2.17: The variation ρ_J of ρ from Figure 2.16 induced by the Jacobi field J . The dashed line segments indicate edges deleted in the construction of ρ_J .

and $[\text{sides}(J(\rho_s)) - (\text{length}(\rho_s) + 2)] \geq 0$ for all $1 \leq s \leq r(J)$, we have

$$\begin{aligned} \text{length}(\rho_J) &\leq 2 + \sum_{s=1}^{r(J)} [\text{sides}(J(\rho_s)) - (\text{length}(\rho_s) + 2)] \\ &= 2 + \sum_{s=1}^{r(J)} \text{length}(\rho_s) [\text{sides}(J(\rho_s)) - (\text{length}(\rho_s) + 2)] \\ &\quad - \sum_{s=1}^{r(J)} [\text{length}(\rho_s) - 1] [\text{sides}(J(\rho_s)) - (\text{length}(\rho_s) + 2)] \end{aligned}$$

We set

$$\Gamma(\rho_s) := [\text{length}(\rho_s) - 1] [\text{sides}(J(\rho_s)) - (\text{length}(\rho_s) + 2)]$$

and remark $\Gamma(\rho_s) \geq 0$. In Forman's discussion this term is neglected. We obtain

$$\begin{aligned} \text{length}(\rho) &\leq \text{length}(\rho_J) \\ &\leq 2 + \sum_{s=1}^{r(J)} \text{length}(\rho_s) [\text{sides}(J(\rho_s)) - (\text{length}(\rho_s) + 2)] - \sum_{s=1}^{r(J)} \Gamma(\rho_s) \\ &= 2 + \sum_{s=1}^k [\text{sides}(J(\beta_s)) - 3] - \sum_{s=1}^{r(J)} \text{length}(\rho_s) [\text{length}(\rho_s) - 1] - \sum_{s=1}^{r(J)} \Gamma(\rho_s), \end{aligned}$$

or, equivalently,

$$0 \leq 2 + \sum_{s=1}^k [\text{sides}(J(\beta_s)) - 4] - \sum_{s=1}^{r(J)} \text{length}(\rho_s) [\text{length}(\rho_s) - 1] - \sum_{s=1}^{r(J)} \Gamma(\rho_s).$$

Choose an arbitrary edge β of ρ . Lemma 2.7.1 tells us that there is a unique Jacobi field along ρ for each 2-face incident to β . In particular, there are precisely d Jacobi fields along ρ . Let us now sum over all Jacobi fields along ρ :

$$\begin{aligned} 0 &\leq \sum_{J \text{ Jacobi field}} \left[2 + \sum_{s=1}^k [\text{sides}(J(\beta_s)) - 4] - \sum_{s=1}^{r(J)} \text{length}(\rho_s) [\text{length}(\rho_s) - 1] - \sum_{s=1}^{r(J)} \Gamma(\rho_s) \right] \\ &\leq 2d + \sum_{J \text{ Jacobi field}} \sum_{s=1}^k [\text{sides}(J(\beta_s)) - 4] - \sum_{J \text{ Jacobi field}} \sum_{s=1}^{r(J)} [2[\text{length}(\rho_s) - 1] + \Gamma(\rho_s)] \\ &= 2d + \sum_{s=1}^k \sum_{\gamma > \beta_s} [\text{sides}(\gamma) - 4] - \sum_{J \text{ Jacobi field}} \sum_{s=1}^{r(J)} [2[\text{length}(\rho_s) - 1] + \Gamma(\rho_s)] \end{aligned}$$

For every $1 \leq s \leq r(J)$ and ρ_s a path from α_1 to α_2 , we have

$$\text{length}(\rho_s) - 1 = |\{\tilde{\alpha} \mid \tilde{\alpha} \text{ a vertex of } \rho_s\} \setminus \{\alpha_1, \alpha_2\}|.$$

For a given path, only one Jacobi field can be constant for two consecutive edges. Hence,

$$\sum_{\substack{J \\ \text{Jacobi field}}} \sum_{s=1}^{r(J)} 2[\text{length}(\rho_s) - 1] = 2 \sum_{s=1}^{k-1} 1,$$

and so we end up with

$$0 \leq 2(d+1) + \sum_{s=1}^k \sum_{\gamma > \beta_s} [\text{sides}(\gamma) - 4] - 2k - \sum_{\substack{J \\ \text{Jacobi field}}} \sum_{s=1}^{r(J)} \Gamma(\rho_s). \quad (2.2)$$

The great challenge is now to obtain weighted version of this inequality that can be partially matched with Inequality (2.1).

Standard Set of Weights: We start with a description in case of a standard set of weights that is already discussed by Forman. If we assume that the weight $\sqrt{\kappa_1 \cdot \kappa_2^p}$ is assigned to each p -cell, then the proper modification is more or less obvious: Multiply Inequality (2.2) by $\frac{1}{\kappa_2}$ and neglect the term that contains $\Gamma(\rho_s)$ to obtain:

$$0 \leq \frac{2(d+1)}{\kappa_2} + \sum_{s=1}^k \sum_{\gamma > \beta_s} \frac{\text{sides}(\gamma) - 4}{\kappa_2} - \frac{2k}{\kappa_2}.$$

This matches perfectly what we obtained in Equation (2.1):

$$0 < ck \leq \sum_{j=1}^k \frac{1}{\kappa_2} + \sum_{j=1}^k \frac{1}{\kappa_2} + \sum_{j=1}^k \sum_{\gamma > \beta_j} \frac{4 - \text{sides}(\gamma)}{\kappa_2} = \sum_{j=1}^k \sum_{\gamma > \beta_j} \frac{4 - \text{sides}(\gamma)}{\kappa_2} + \frac{2k}{\kappa_2}.$$

Adding these two inequalities yields

$$0 < ck \leq \frac{2(d+1)}{\kappa_2},$$

that is, we obtain $\frac{2(d+1)}{c\kappa_2}$ as an upper bound for the diameter of the simple d -manifold, since k is the length of an arbitrary shortest path.

We now discuss some examples.

A first example: As first example we consider the boundary of a $(d+1)$ -dimensional cube which is a d -manifold. To warm up, we study two different sets of weights: A standard set of weights and the geometric set of weights.

As the standard set of weights is concerned, we make the trivial choice where all weights equal 1. It is easy to see that the Ricci curvature is constant and equals 2. Hence we derive $k \leq d+1$ as upper bound for the diameter of the boundary a $(d+1)$ -cube.

If we assign the geometric weights to the cells of the 2-skeleton of the $(d+1)$ -cube, that is, $w_{0,\alpha}^2 = \frac{1}{\deg(\alpha)} = \frac{1}{d+1}$ ($\deg(\alpha)$ denotes the degree of vertex α), $w_{1,\beta} = 1$, and

$w_{2,\gamma}^2 = \text{sides}(\gamma) = 4$ ($\text{sides}(\gamma)$ is the number of edges in the boundary of γ), then the combinatorial Ricci curvature computes for each edge β

$$\text{Ric}(\beta) = \sum_{\alpha < \beta} \frac{w_{0,\alpha}^2}{w_{1,\beta}^2} + \sum_{\gamma > \beta} [4 - \text{sides}(\gamma)] \frac{w_{1,\beta}^2}{w_{2,\gamma}^2} = \sum_{\alpha < \beta} \frac{1}{d+1} = \frac{2}{d+1}.$$

The Ricci curvature is therefore constant and equals $\frac{2}{d+1} > 0$. This implies that in Inequality (2.1) holds equality. Moreover, we do not need this Equation, since $(\text{sides}(\gamma) - 4) = 0$ for each 2-face γ . Since $\Gamma(\rho_s) = 0$, Inequality (2.2) becomes

$$0 \leq 2(d+1) - 2k.$$

Hence we obtain the same upper bound as in case of a standard set of weights: $k \leq d+1$. **A second example:** The boundary of a dodecahedron depicted in Figure 2.13 is a simple 2-manifold and an example that the Forman's original approach is not applicable to convex polytopes in general. All edges are Ricci-flat if we restrict to a standard set of weights. If we consider the geometric set of weights, the situation becomes better at first sight: Every edge β is positively Ricci curved:

$$\text{Ric}(\beta) = \frac{2}{3} - \frac{2}{5} = \frac{4}{15} \quad \text{for every edge } \beta.$$

As in the first example, we neglect the term of Inequality 2.2 that contains $\Gamma(\rho_s)$, multiply by $\frac{1}{5}$ (since $\text{sides}(\gamma) = 5$ for each 2-face γ), and add the resulting inequality to Inequality 2.1 to obtain

$$0 < \frac{4}{15}k \leq \frac{2(d+1)}{5} + \frac{2}{3}k - \frac{2}{5}k = \frac{2(d+1)}{5} + \frac{4}{15}k.$$

Unfortunately, the terms that contain k cancel each other. But fortunately enough, we can sharpen this inequality a bit if we take the term that contains $\Gamma(\rho_s)$ into account. This yields

$$0 \leq \frac{2(d+1)}{5} - \sum_{\substack{J \\ \text{Jacobi field}}} \sum_{s=1}^{r(J)} \frac{\Gamma(\rho_s)}{5}.$$

But $\sum \sum \Gamma(\rho_s)$ can be computed as follows: A shortest path has to leave a pentagon after it travelled along at most two of its sides, otherwise there is a shorter path. Hence $0 < \text{length}(\rho_s) \leq 2$ for each s . Moreover, at every *inner vertex* of ρ , that is, any vertex of ρ that is different from the endpoints, there is a Jacobi field along ρ that is constant. Hence,

$$\sum_{\substack{J \\ \text{Jacobi field}}} \sum_{s=1}^{r(J)} \Gamma(\rho_s) = k - 1.$$

Using this equality, we get $k \leq 2d+3 = 7$ which is certainly not sharp since the diameter of the dodecahedron is 5. The Hirsch bound $n - d$ is $12 - 3 = 9$.

A third example: The boundary of the polar of a cyclic 3-polytope on six vertices is our third example and is visualised in Figure 2.12. This example is more interesting than the preceeding ones since Ricci curvature will not turn out to be constant. Again, Forman's original approach is not applicable, since one edge has vanishing combinatorial Ricci curvature if we assign a standard set of weights. Instead, we choose the following weights: $w_{0,\alpha}^2 = 1$ for all vertices α , $w_{1,\beta}^2 = 1$ for all edges β , and $w_{2,\gamma}^2 = r$ with $r > 1$ for all 2-cells γ . If we compute the Ricci curvatures with respect to this set of weights, we obtain one edge of smallest curvature $c_1 = 2 - \frac{2}{r}$. All other edges have larger curvature. The second smallest curvature is $c_2 = 2 - \frac{1}{r}$. We use the fact that there is only one edge of minimal curvature to modify Inequality 2.1:

$$0 < c_1 k \leq c_1 k + \frac{1}{r}(k-1) \leq 2k + \sum_{j=1}^k \sum_{\gamma > \beta_j} \frac{4 - \text{sides}(\gamma)}{r}.$$

Since $\Gamma(\rho_s) \geq 0$ we obtain from Inequality 2.2

$$0 \leq 2(d+1) + \sum_{s=1}^k \sum_{\gamma > \beta_s} [\text{sides}(\gamma) - 4] - 2k.$$

Hence we end up with

$$0 < c_1 k + \frac{1}{r}(k-1) \leq \frac{2(d+1)}{r} + 2k(1 - \frac{1}{r}) = \frac{2(d+1)}{r} + c_1 k.$$

Since we may assume $k > 1$, we obtain

$$0 < k-1 \leq 2(d+1),$$

which implies $k \leq 7$ since the boundary of a 3-polytope is a 2-manifold. The Hirsch bound computes as $6 - 3 = 3$.

This method does not extend to polars of cyclic 3-polytopes on n vertices in the “natural” way if $n > 6$. We recollect the fact that these polytopes are wedges over $(n-1)$ -gons. If we choose the weights $w_{0,\alpha}^2 = 1$, $w_{1,\beta}^2 = 1$, and $w_{2,\gamma}^2 = (n-5)r$ with $r > 1$, we obtain again $c_1 = 2 - \frac{2}{r}$ and $c_2 = 2 - \frac{1}{r}$, where c_1 is attained once. The same reasoning as above yields

$$0 < c_1 k + \frac{1}{r}(k-1) \leq \frac{2(d+1)}{(n-5)r} + 2k(1 - \frac{1}{r}) - 2k \frac{6-n}{(n-5)r},$$

which in turn can be reduced to $(7-n)k \leq 2d+n-3$ which gives a lower bound for k if and only if $n < 7$. A different choice of weights might yield lower bounds for larger n , but no choice is known.

A fourth example: As last example, we consider a family of simple 2-tori made of hexagons. These tori are obtained by identifying the boundary of an $(m \times n)$ -patch of hexagons as indicated in Figure 2.18. If we weight the vertices by w_0 , the edges by w_1 and

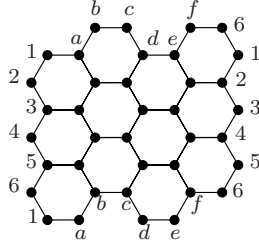


Figure 2.18: The hexagonal torus obtained from an (4×3) -patch of hexagons by identifying the boundary as indicated. The diameter of the resulting torus depends on the number of hexagons used in the patch.

the 2-cells by w_2 , then the combinatorial Ricci curvature is constant for each edge β and equals

$$c := \text{Ric}(\beta) = 2 \left(\frac{w_0}{w_1} - 2 \cdot \frac{w_1}{w_2} \right).$$

Without loss of generality we may assume $w_1 = 1$. Hence $\text{Ric} > 0$ if and only if $w_0 w_2 > 2$. We now need an estimate for

$$\sum_{J \text{ Jacobi field}} \sum_{s=1}^{r(J)} \Gamma(\rho_s) = \sum_{J \text{ Jacobi field}} \sum_{s=1}^{r(J)} [\text{length}(\rho_s) - 1] [\text{sides}(J(\rho_s)) - \text{length}(\rho_s) - 2].$$

To this respect, we observe that a Jacobi field along a shortest path can at most be of length 3, otherwise there is a shortcut. Hence,

$$\begin{aligned} \sum_{J \text{ Jacobi field}} \sum_{s=1}^{r(J)} \Gamma(\rho_s) &= \sum_{J \text{ Jacobi field}} \sum_{s=1}^{r(J)} [\text{length}(\rho_s) - 1] [\text{sides}(J(\rho_s)) - \text{length}(\rho_s) - 2] \\ &= \sum_{J \text{ Jacobi field}} \sum_{s=1}^{r(J)} [\text{length}(\rho_s) - 1] \\ &= (k - 1) \end{aligned}$$

We obtain from Inequality 2.2

$$\begin{aligned} 0 &\leq 2(d + 1) + \sum_{s=1}^k \sum_{\gamma > \beta_s} [\text{sides}(\gamma) - 4] - 2k - (k - 1) \\ &= 2(d + 1) + 1 + \sum_{s=1}^k \sum_{\gamma > \beta_s} [\text{sides}(\gamma) - 4] - 3k \\ &= 2d + 3 + k, \end{aligned}$$

that is, $-k < 2d + 3$ which does not yield an upper bound for k . Since the Ricci curvature is constant, Inequality 2.1 does not help to give an upper bound. So either some modifications

make this method applicable to the hexagonal torus or not. One approach could be to find a more suitable set of weights. In any case, such a modification has to reflect the number mn of hexagons used, since the diameter certainly varies with the number of facets.

Part II

Topology and Combinatorics

INTRODUCTION

During the last 25 years, topological methods have been successfully applied to solve difficult problems in combinatorics and geometry. Famous examples are the necklace problem solved by Alon [2], the topological Tverberg theorem proved by Bárány, Shlosman, and Szücs [8] which generalises Radon’s theorem, and lower bounds for the chromatic number of graphs and hypergraphs. Common to all these problems is the use of some version of the Borsuk–Ulam theorem at some point. More advanced methods of algebraic topology (and algebraic geometry) have been employed in combinatorics, for example the use of the hard Lefschetz theorem by Stanley to prove the Erdős–Moser conjecture [59] and to show the necessity in the characterisation of f -vectors of simplicial convex polytopes [58, 60]. A different direction is the use of Stiefel–Whitney classes by Babson and Kozlov [5, 4, 6] in the proof of a lower bound of the chromatic number of a graph that uses *Hom-complexes*. However, the area took off with Lovász’ seminal paper [46] from 1978 where he proved that the connectivity of the neighbourhood complex $N(G)$ of a graph G can be used to establish a lower bound of its chromatic number $\chi(G)$. He used this bound to prove Kneser’s conjecture [39], which dates back to 1955:

The chromatic number of the Kneser graph $KG\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right)$ is $n-2k+2$ for $n>2k-1$.

The Kneser graph $KG\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right)$ of the k -subsets $\begin{smallmatrix} [n] \\ k \end{smallmatrix}$ of the n -set $[n]$ has the k -subsets of $[n]$ as nodes and two nodes form an edge if they are disjoint. For example, the Petersen graph is the Kneser graph $KG\left(\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right)$ of the 2-subsets of a 5-set, see Figure 2.19.

The pattern of the *topological method* initiated by Lovász is easily described. Firstly, associate a topological space together with a free or fixed-point-free group action to the given combinatorial object, e.g., a simplicial complex to a given graph. Secondly, invariants such as the “ \mathbb{Z}_r -index”, the dimension of the first non-vanishing reduced homology with \mathbb{Z}_r -coefficients, or the connectivity of the associated simplicial complex are then related to the combinatorial problem, e.g., as a lower bound of the chromatic number of the graph. Lovász

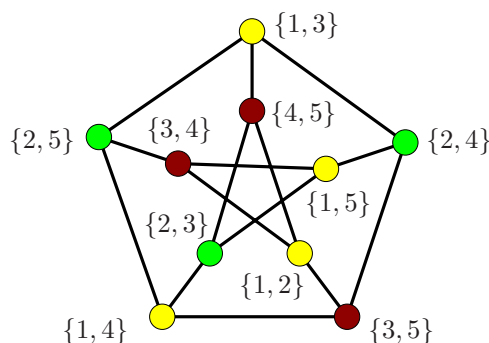


Figure 2.19: The Kneser graph $KG\left(\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right)$ of the 2-subsets of $[5]$ is the Petersen graph. The graph is coloured with three colours by a *greedy-type colouring*, that is, colour all nodes that contain 1 with the first colour, all nodes that are not coloured yet and that contain 2 with the second colour and the remaining nodes with the third colour.

used a free \mathbb{Z}_2 -action which is a “natural” group action for graphs. But other problems require different group actions. Troubles often occur if r is not the power of a prime. As a consequence, some theorems proved by the topological method remain conjectures if the statement is reformulated for r not the power of a prime. The most famous open problem in this respect is the topological Tverberg conjecture: Bárány, Shlosman, and Szűcs [8] were only able to prove the prime case using a free \mathbb{Z}_p -action. Later Özaydin [54] and Volovikov [64] generalised this to the case of prime-powers using a fixed-point free $(\mathbb{Z}_p)^t$ -action. Similarly, a lower bound for the chromatic number of r -uniform hypergraphs is only achieved if r is the power of a prime. But if one restricts to subclasses of this problem, combinatorial reasoning may circumvent the topological problems. A lower bound for the chromatic number of an r -uniform Kneser hypergraph can be given for all r in purely combinatorial terms by the colourability defect. This is achieved as follows: Firstly, it can be shown that the colourability defect is a lower bound for the chromatic number if r is prime. Then a combinatorial argument shows by induction that the colourability defect is in fact a lower bound for all r . This argument was given by Alon, Frankl, and Lovász for the first time in [3] and has been modified to a more general setting by Kříž [41]. Sarkaria [56] and Ziegler [69] again generalise this method, but unfortunately there is a gap in their arguments. We discuss the problems involved at the end of Chapter 4.

Chapter 3 stays within the category of graphs. We introduce the concept of the *shore subdivision* of a simplicial complex which has independently been used by de Longueville [22] to give an elegant proof that Bier spheres are in fact spheres as their name indicates. The main result of this chapter is that we find a (non-canonical) \mathbb{Z}_2 -isomorphic copy of the complex Lovász used in the shore subdivision of a boxcomplex and show that the Lovász complex is a strong deformation retract of this box complex. In contrast to the complex $L(G)$ of Lovász, the box complex has a functorial property, that is, a graph homomorphism $f : G \rightarrow H$ induces a simplicial \mathbb{Z}_2 -map $B(f) : B(G) \rightarrow B(H)$. This makes conceptually easy proofs possible. Walker [65] constructed a non-canonical \mathbb{Z}_2 -map $\phi : L(G) \rightarrow L(H)$ from a graph homomorphism f , but his construction is rather complicated. We give a simpler description of such a map once the \mathbb{Z}_2 -isomorphic copies of the Lovász complexes involved are chosen. The chapter ends with an upper bound of the topological lower bound of the chromatic number of a graph. We show that if a graph does not contain a complete bipartite subgraph of type $K_{\ell,m}$, then the lower bound obtained by these topological methods is at most $\ell + m - 3$. This extends a result of Walker [65] who proved the case $\ell = m = 2$. As a consequence, the topological lower bound can become arbitrarily bad. Chapter 3 is joint work with Péter Csorba, Ingo Schurr, and Arnold Waßmer.

Chapter 4 is devoted to generalised Kneser colouring problems. These problems can be formulated as colouring problems of two versions of r -uniform Kneser hypergraphs. The two concepts differ with respect to possible hyperedges. We define r -uniform hypergraphs *without* multiplicities, where a hyperedge must contain r distinct nodes, and r -uniform hypergraphs *with* multiplicities, where a hyperedge is allowed to contain multiple copies of a node and contains at least two different nodes. The classical notion of a hypergraph is a hypergraph without multiplicities according to this terminology. The first result we prove a generalisation of a result of Alon, Frankl, and Lovász on r -uniform hypergraphs

without multiplicities to r -uniform hypergraphs with multiplicities: The chromatic number of such a hypergraph has a lower bound achieved by topological methods. We then show by example that a result of Sarkaria [56] and of Ziegler [69] does not hold if we consider r -uniform generalised Kneser hypergraphs *without* multiplicities and give a new proof of Ziegler's result that is inspired by Matoušek's [49] proof of a result of Kříž [40, 41]. The inductive argument to extend the result to non-prime r given by Sarkaria and Ziegler is not complete. We discuss the gaps and solve some special cases.

CHAPTER 3

CHROMATIC NUMBERS OF GRAPHS

INTRODUCTION

In this chapter we present a subdivision technique and use it to show that the complex $L(G)$ of a graph G , which Lovász used (and which we call Lovász complex for that reason), has an interpretation as \mathbb{Z}_2 -deformation retract of the box complex $B(G)$, as described by Matoušek and Ziegler [51]. We explicitly realise the Lovász complex $L(G)$ as a \mathbb{Z}_2 -subcomplex of the *shore subdivision* $\text{ssd}(B(G))$. This realisation depends on the choice of a linear order and yields the *halved doubled Lovász complex* $\text{HDL}(G)$.

The advantage of the box complex is its functorial property: For every graph homomorphism $f : G \rightarrow H$ one obtains an induced simplicial \mathbb{Z}_2 -map $B(f) : B(G) \rightarrow B(H)$. This functorial property gives elegant conceptual proofs: A colouring of a graph G with m colours is a graph homomorphism from G into the complete graph K_m on m vertices, and the index of $B(K_m)$ is well-known. The Lovász complex does not behave that nicely: There is no canonical map between $L(G)$ and $L(H)$ known that is canonically induced from a graph homomorphism. Walker [65] constructed a \mathbb{Z}_2 -map $\varphi : \|L(G)\| \rightarrow \|L(H)\|$ which is not canonical. This construction is rather involved. The realisation $\text{HDL}(G)$ of the Lovász complex $L(G)$ as a \mathbb{Z}_2 -subcomplex of $\text{ssd}(B(G))$ and the functorial property of the box complex can be used to construct a (non-canonical) \mathbb{Z}_2 -map $\text{HDL}(f) : \text{HDL}(G) \rightarrow \text{HDL}(H)$ in a straight-forward way. The construction is not canonical, since choices are involved to realise the Lovász complexes as “halved doubled Lovász complexes”.

The box complex $B(G)$ of G yields a lower bound for the chromatic number $\chi(G)$:

$$\chi(G) \geq \text{ind}(B(G)) + 2.$$

It is known that this topological bound can get arbitrarily bad: Walker [65] shows that if a graph G does not contain a $K_{2,2}$ as a (not necessarily induced) subgraph, then the associated invariant yields 3 as lower bound for the chromatic number $\chi(G)$. We generalise this result to the following statement: If G does not contain a completely bipartite graph $K_{\ell,m}$ then the index of the box complex $B(G)$ is bounded by $\ell + m - 3$.

The chapter is organised as follows. We summarise basic definitions and results on graphs, simplicial complexes, \mathbb{Z}_2 -spaces and their indices, neighbourhood complexes, Lovász complexes, and the box complexes in Section 3.1. The *shore subdivision*, the *doubled Lovász complex*, and the *halved doubled Lovász complex* are defined in Section 3.2. This section ends with an example to illustrate all these complexes. In Section 3.3 we prove that the Lovász complex is \mathbb{Z}_2 -isomorphic to the halved doubled Lovász complex and that the halved

doubled Lovász complex is a strong \mathbb{Z}_2 -deformation retract of the box complex. Section 3.3 ends with a construction of a \mathbb{Z}_2 -map $\mathbf{HDL}(f) : \mathbf{HDL}(G) \longrightarrow \mathbf{HDL}(H)$ that is induced from a graph homomorphism $f : G \longrightarrow H$. We close this chapter with Section 3.4 where we prove the upper bound for the topological lower bound.

This chapter is joint work with Péter Csorba, Ingo Schurr, and Arnold Waßmer.

3.1 PRELIMINARIES

In this section we recall some basic facts of graphs, simplicial complexes, and \mathbb{Z}_2 -actions. Moreover, we define the classical complexes Lovász used in his proof of Kneser's conjecture as well as a version of a box complex. A simple example in Section 3.2 illustrates these (and other) complexes. The interested reader is referred to Matoušek [50] and Björner [13] for details.

Graphs: We assume any graph G to be finite, simple, connected, and undirected. Hence, G is given by a finite set $V(G)$ of *nodes* (we use *vertices* for associated complexes) and a set of *edges* $E(G) \subseteq \binom{V(G)}{2}$. A *graph homomorphism* f between two graphs G and H is a map that maps nodes to nodes and edges to edges. A *proper graph colouring with n colours* is a homomorphism $c : G \rightarrow K_n$, where K_n is the complete graph on n nodes. The *chromatic number* $\chi(G)$ of G is the smallest n such that a proper graph colouring of G with n colours exists. The *neighbourhood* $N(u)$ of a node $u \in V(G)$ is the set of all nodes adjacent to u . For a set of nodes $A \subseteq V(G)$, a node v is in the *common neighbourhood* $CN(A)$ of A if v is adjacent to all $a \in A$; we define $CN(\emptyset) := V(G)$. For $A \subseteq B \subseteq V(G)$, the common neighbourhood relation satisfies the following elementary identities:

$$\begin{aligned} A \cap CN(A) &= \emptyset, \\ CN(B) &\subseteq CN(A), \\ A &\subseteq CN^2(A), \text{ and} \\ CN(A) &= CN^3(A). \end{aligned}$$

Because of the last identity we call CN^2 a *closure operator*. For two disjoint sets of nodes $A, B \subseteq V(G)$ we define $G[A; B]$ as the (not necessarily induced) subgraph of G with node set $V(G[A; B]) = A \cup B$ and all edges $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. For a given node set A , the set $CN(A)$ is the inclusion-maximal set B such that $G[A; B]$ is complete bipartite.

Constructions for simplicial complexes: We denote the vertex set of an *abstract simplicial complex* K by $V(K)$, and its *barycentric subdivision* by $sd(K)$. Another important construction in the category of simplicial complexes is the *join operation*. For its definition, we introduce the following notation. For sets A, B we define

$$A \uplus B := \{(a, 0) \mid a \in A\} \cup \{(b, 1) \mid b \in B\}.$$

For two simplicial complexes K and L the join $K * L$ is defined as

$$K * L := \{F \uplus G \mid F \in K \text{ and } G \in L\}.$$

A *simplicial map* between the simplicial complexes K and L is a map $f : V(K) \rightarrow V(L)$ that maps simplices to simplices. An isomorphism of abstract simplicial complexes is a simplicial map with a simplicial inverse. Every abstract simplicial complex K can be realised as a topological space $\|K\|$ in \mathbb{R}^d for some d . Such realisations yield a realisation $\|f\|$ of a simplicial map.

\mathbb{Z}_2 -spaces and \mathbb{Z}_2 -index: A \mathbb{Z}_2 -space is a topological space X with a homeomorphism $\nu : X \rightarrow X$ that is self-inverse and free, i.e., has no fixed points. The map ν is called *free \mathbb{Z}_2 -action*. An important example of a \mathbb{Z}_2 -space is the d -sphere S^d together with the antipodal map $\nu(x) = -x$. A continuous map f between \mathbb{Z}_2 -spaces (X, ν) and (Y, μ) is \mathbb{Z}_2 -equivariant (or a \mathbb{Z}_2 -map for simplicity) if f commutes with the \mathbb{Z}_2 -actions, that is, $f \circ \nu = \mu \circ f$. A *simplicial \mathbb{Z}_2 -space* is a pair (K, ν) , where K is a simplicial complex and $\nu : K \rightarrow K$ a simplicial map such that $\|\nu\|$ is a free \mathbb{Z}_2 -action on $\|K\|$. A *simplicial \mathbb{Z}_2 -equivariant* map f is a simplicial map between two simplicial \mathbb{Z}_2 -spaces that commutes with the simplicial \mathbb{Z}_2 -actions. A \mathbb{Z}_2 -isomorphism of \mathbb{Z}_2 -spaces is an isomorphism that respects the \mathbb{Z}_2 -action.

The *index* of a \mathbb{Z}_2 -space (X, ν) is the smallest d such that there is a \mathbb{Z}_2 -map $f : X \rightarrow S^d$, i.e., $f \circ \nu = -f$. The Borsuk–Ulam theorem states that there is no antipodal continuous mapping $f : S^d \rightarrow S^{d-1}$. Hence it provides the index for spheres: $\text{ind}(S^d) = d$. Since the \mathbb{Z}_2 -actions are usually canonical, we often refer to a \mathbb{Z}_2 -space K without explicit reference to ν .

Chain Notation: We denote by \mathcal{A} a chain $A_1 \subset \dots \subset A_p$ of subsets of the nodes $V(G)$ of a graph G and by \mathcal{B} a chain $B_1 \subset \dots \subset B_q$ of subsets of $V(G)$. For $1 \leq t \leq p$ we denote by $\mathcal{A}_{\leq t}$ the chain $A_1 \subset \dots \subset A_t$. A similar notation is used for $\mathcal{A}_{\geq t}$. Chains \mathcal{A}, \mathcal{B} satisfying $A_p \subseteq B_1$ can be concatenated to a new chain $\mathcal{A} \sqsubseteq \mathcal{B}$:

$$\mathcal{A} \sqsubseteq \mathcal{B} := A_1 \subset \dots \subset A_p \subseteq B_1 \subset \dots \subset B_q,$$

where we omit A_p in case $A_p = B_1$. If a map f preserves (resp. reverses) orders, we write $f(\mathcal{A})$ for $f(A_1) \subseteq \dots \subseteq f(A_p)$ (resp. $f(A_p) \subseteq \dots \subseteq f(A_1)$).

Neighbourhood Complex: The *neighbourhood complex* $N(G)$ of a graph G has the vertex set $V(G)$ and the sets $A \subseteq V(G)$ with $\text{CN}(A) \neq \emptyset$ as simplices.

Lovász Complex: In general $N(G)$ is not a \mathbb{Z}_2 -space. However, the neighbourhood complex can be retracted to a \mathbb{Z}_2 -subspace, the *Lovász complex*. This complex $L(G)$ is the subcomplex of $\text{sd}(N(G))$ induced by the vertices that are fixed points of CN^2 . The retraction is induced from mapping a vertex $A \in N(G)$ to $\text{CN}^2(A)$. The Lovász complex is

$$L(G) = \{\mathcal{A} \mid \mathcal{A} \text{ a chain of node sets of } G \text{ with } \mathcal{A} = \text{CN}^2(\mathcal{A})\}$$

which is a \mathbb{Z}_2 -space with \mathbb{Z}_2 -action CN .

Box Complex: Different versions of box complexes are described by Alon, Frankl, and Lovász [3], Sarkaria [56], Kříž [40], and Matoušek and Ziegler [51]. The *box complex* $B(G)$ of a graph G we need is defined by Matoušek and Ziegler as

$$\begin{aligned} B(G) &:= \{A \uplus B \mid A, B \in N(G) \text{ and } G[A; B] \text{ is complete bipartite}\} \\ &= \{A \uplus B \mid A, B \in N(G), A \subseteq \text{CN}(B), \text{ and } B \subseteq \text{CN}(A)\}. \end{aligned}$$

The vertex set of the box complex can be partitioned as follows:

$$V_1 := \{\{v\} \uplus \emptyset \mid v \in V(G)\} \quad \text{and} \quad V_2 := \{\emptyset \uplus \{v\} \mid v \in V(G)\}.$$

The subcomplexes of $\mathbf{B}(G)$ induced by V_1 and V_2 are disjoint subcomplexes of $\mathbf{B}(G)$ that are both isomorphic to the neighbourhood complex $\mathbf{N}(G)$. We refer to these two copies as *shores* of the box complex. The box complex is endowed with a \mathbb{Z}_2 -action ν which interchanges the shores.

3.2 SHORE SUBDIVISION AND USEFUL SUBCOMPLEXES

We introduce the general concept of a *shore subdivision* of a simplicial complex and define the *doubled Lovász complex* $\mathbf{DL}(G)$ and *halved doubled Lovász complex* $\mathbf{HDL}(G)$ of a graph G . The name of the latter traces back to the fact that $\mathbf{HDL}(G)$ has half as many vertices as $\mathbf{DL}(G)$ and that $\mathbf{DL}(G)$ has two copies of the Lovász complex $\mathbf{L}(G)$ as “shores”. We end this section with an easy example to illustrate all complexes defined so far.

Shore Subdivision: Given a simplicial complex \mathbf{K} and a partition $V_1 \sqcup V_2$ of its vertex set, we call the simplicial subcomplexes \mathbf{K}_1 and \mathbf{K}_2 induced by the vertex sets V_1 and V_2 its *shores*. The *shore subdivision* of \mathbf{K} is

$$\text{ssd}(\mathbf{K}) := \{\text{sd}(\sigma \cap \mathbf{K}_1) * \text{sd}(\sigma \cap \mathbf{K}_2) \mid \sigma \in \mathbf{K}\}.$$

We apply this definition to the shores of the box complex to obtain the shore subdivision $\text{ssd}(\mathbf{B}(G))$ of the box complex $\mathbf{B}(G)$. The vertices of $\text{ssd}(\mathbf{B}(G))$ are of type $A \uplus \emptyset$ and $\emptyset \uplus A$, where $\emptyset \neq A \subset V(G)$ with $\text{CN}(A) \neq \emptyset$. A simplex of $\text{ssd}(\mathbf{B}(G))$ is denoted by $\mathcal{A} \uplus \mathcal{B}$ (the simplex spanned by the vertices $A \uplus \emptyset$ and $\emptyset \uplus B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$). The shore subdivision of the box complex $\mathbf{B}(G)$ is endowed with a natural \mathbb{Z}_2 -action induced from the \mathbb{Z}_2 -action of $\mathbf{B}(G)$ that interchanges the shores.

Doubled Lovász Complex: The map $\text{cn}^2 : \text{ssd}(\mathbf{B}(G)) \longrightarrow \text{ssd}(\mathbf{B}(G))$ defined on the vertices by

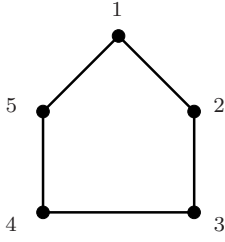
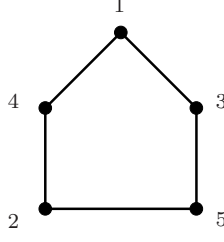
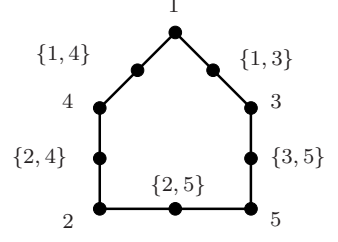
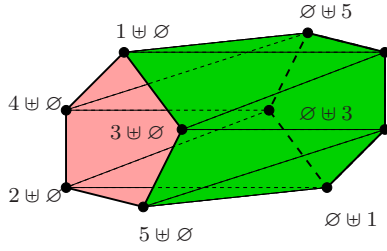
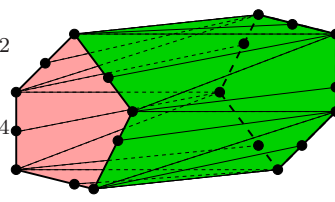
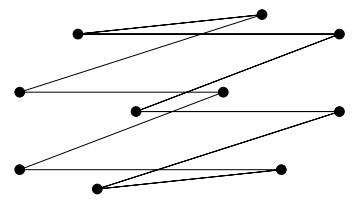
$$\text{cn}^2(A \uplus \emptyset) := \text{CN}^2(A) \uplus \emptyset \quad \text{and} \quad \text{cn}^2(\emptyset \uplus A) := \emptyset \uplus \text{CN}^2(A)$$

is simplicial (the shores of $\text{ssd}(\mathbf{B}(G))$ are copies of $\text{sd}(\mathbf{N}(G))$ and CN^2 is a simplicial map on $\text{sd}(\mathbf{N}(G))$) and \mathbb{Z}_2 -equivariant. We refer to the image $\text{Im } \text{cn}^2$ as the *doubled Lovász complex* $\mathbf{DL}(G)$. It is

$$\mathbf{DL}(G) = \left\{ \mathcal{A} \uplus \mathcal{B} \mid \begin{array}{l} \mathcal{A}, \mathcal{B} \in \mathbf{L}(G), \\ G[A; B] \text{ is completely bipartite for all } A \in \mathcal{A}, B \in \mathcal{B} \end{array} \right\}.$$

The \mathbb{Z}_2 -action is the induced \mathbb{Z}_2 -action of $\text{ssd}(\mathbf{B}(G))$. A copy of the Lovász complex can be found on each shore of $\mathbf{DL}(G) \subseteq \text{ssd}(\mathbf{B}(G))$, but the \mathbb{Z}_2 -actions of these copies map vertices of one shore to vertices of the same shore.

Halved Doubled Lovász Complex: We partition the vertex set of the doubled Lovász complex $\mathbf{DL}(G)$ into pairs of type $\{A \uplus \emptyset, \emptyset \uplus \text{CN}(A)\}$ to define a simplicial \mathbb{Z}_2 -map j on $\mathbf{DL}(G)$. The map j will be defined by specifying one vertex of each pair as image of both vertices under j . We call this specified vertex the *smaller vertex of the pair*. Before

Figure 3.1: The graph C_5 .Figure 3.2: The neighbourhood complex $N(C_5)$.Figure 3.3: The Lovász complex $L(C_5)$.Figure 3.4: The box complex $B(C_5)$.Figure 3.5: The shore subdivision $\text{ssd}(B(C_5)) = DL(C_5)$.Figure 3.6: The halved doubled Lovász complex $HDL(C_5)$.

we define a partial order on $V(DL(G))$, we refine the partial order on $V(DL(G))$ given by cardinality using the lexicographic order:

$$A \prec B \quad :\Longleftrightarrow \quad \begin{cases} |A| < |B| \text{ or} \\ |A| = |B| \text{ and } A <_{\text{lex}} B. \end{cases}$$

In fact, any refinement would work in the following. A partial order on the vertices of the doubled Lovász complex $DL(G)$ is now obtained by:

$$A \uplus \emptyset \prec \emptyset \uplus CN(A) \quad :\Longleftrightarrow \quad A \prec CN(A).$$

We define the map j on the vertices of $DL(G)$ using this partial order:

$$j(A \uplus \emptyset) := \min_{\prec} \{A \uplus \emptyset, \emptyset \uplus CN(A)\}, \quad \text{and} \quad j(\emptyset \uplus B) := \min_{\prec} \{\emptyset \uplus B, CN(B) \uplus \emptyset\}.$$

Since the image $\text{Im } j$ has half as many vertices as $DL(G)$, we refer to $\text{Im } j$ as *halved doubled Lovász complex* $HDL(G)$; its \mathbb{Z}_2 -action is induced from $\text{ssd}(B(G))$ or $DL(B(G))$.

An example: The neighbourhood complex $N(C_5)$ of the 5-cycle C_5 is the 5-cycle; its Lovász complex $L(C_5)$ is the 10-cycle C_{10} , compare Figures 3.1, 3.2, and 3.3. The box complex $B(C_5)$ consists of two copies of $N(C_5)$ (the two shores) such that simplices of different shores are joined if and only if the corresponding node sets are common neighbours of each other. The box complex $B(C_5)$ is visualised in Figure 3.4. The shore subdivision $\text{ssd}(B(C_5))$ is a subdivision of the box complex induced from a barycentric subdivision of the shores,

compare Figure 3.5. The map cn^2 maps a vertex of $\text{ssd}(B(C_5))$ to the common neighbourhood of its common neighbourhood. In our example, every vertex is mapped to itself, hence $\text{ssd}(B(C_5)) = \text{DL}(C_5)$. The partitioning of the vertex set of $\text{DL}(C_5)$ into pairs of type $(A \uplus \emptyset, \emptyset \uplus \text{CN}(A))$ can be visualised by edges of $\text{DL}(C_5)$ that connect singletons from one shore with 2-element sets from the other. The refined lexicographic order determines the image of such an edge under j : the smaller vertex is the singleton. Hence the map j collapses all edges of type $(A \uplus \emptyset, \emptyset \uplus \text{CN}(A))$, which yields the halved doubled Lovász complex $\text{HDL}(G)$, see Figure 3.6. The maps f_i that will be introduced in Section 3.3 are these collapses and they will be used to show that $L(G)$ is a \mathbb{Z}_2 -deformation retract of $\text{ssd}(B(G))$.

3.3 $L(G)$ AS A \mathbb{Z}_2 -DEFORMATION RETRACT OF $B(G)$

Now we show that the halved doubled Lovász complex $\text{HDL}(G)$ is \mathbb{Z}_2 isomorphic to the original Lovász complex $L(G)$ of a given graph G and that $\text{HDL}(G)$ is a strong deformation retract of the shore subdivision $\text{ssd}(B(G))$. We end this chapter with a sketch how an induced map $\text{HDL}(f) : \text{HDL}(G) \rightarrow \text{HDL}(H)$ can be obtained from a graph homomorphism f .

Theorem 3.3.1. The Lovász complex $L(G)$ and the halved doubled Lovász complex $\text{HDL}(G)$ are \mathbb{Z}_2 -isomorphic.

Proof. First we have $|V(L(G))| = |V(\text{HDL}(G))|$, since each shore of $\text{DL}(G)$ is isomorphic (but not \mathbb{Z}_2 -isomorphic) to $L(G)$. To define a simplicial \mathbb{Z}_2 -map $f : L(G) \rightarrow \text{HDL}(G)$, we partition $V(L(G))$ into

$$S := \left\{ A \mid A \in V(L(G)) \text{ and } j(A \uplus \emptyset) = A \uplus \emptyset \right\} \quad \text{and} \quad J := \left\{ A \mid A \in V(L(G)) \text{ and } j(A \uplus \emptyset) = \emptyset \uplus \text{CN}(A) \right\},$$

(where “ S ” and “ J ” denote the vertices that *Stay* fixed or *Jump* to their neighbour), and set

$$f(A) := \begin{cases} A \uplus \emptyset & \text{if } A \in S \\ \emptyset \uplus \text{CN}(A) & \text{if } A \in J. \end{cases}$$

This map is a bijection between the vertex sets $V(L(G))$ and $V(\text{HDL}(G))$ and \mathbb{Z}_2 -equivariant by definition, since the \mathbb{Z}_2 -action of $L(G)$ is CN and the \mathbb{Z}_2 -action of $\text{HDL}(G)$ maps every vertex of one shore to its copy on the other shore. We now show that f is simplicial and surjective. For simpliciality, consider a simplex \mathcal{A} in $L(G)$. Let t denote the largest index k such that A_k is mapped onto the first shore. The image of \mathcal{A} under f is $\mathcal{A}_{\leq t} \uplus \text{CN}(\mathcal{A}_{\geq t+1})$. This is a simplex, since $G[A_t; \text{CN}(A_{t+1})]$ is completely bipartite. For surjectivity consider a simplex $\mathcal{A} \uplus \mathcal{B}$ of $\text{HDL}(G)$, i.e., $G[A_p; B_q]$ is completely bipartite. This simplex is the image of the simplex $\mathcal{A} \sqsubseteq \text{CN}(\mathcal{B})$ of $L(G)$. \square

Theorem 3.3.2. The halved doubled Lovász complex $\text{HDL}(G)$ is a strong \mathbb{Z}_2 -deformation retract of the box complex $B(G)$.

Proof. First we observe that $\|\mathbf{DL}(G)\|$ is a strong \mathbb{Z}_2 -deformation retract of $\|\mathbf{ssd}(\mathbf{B}(G))\| = \|\mathbf{B}(G)\|$. This follows from the fact that a closure operator induces a strong deformation retraction from its domain to its image; see Björner [13, Corollary 10.12 and following remark]. Explicitly, this map is obtained by sending each point $p \in \|\mathbf{ssd}(\mathbf{B}(G))\|$ towards $\|\mathbf{CN}^2\|(p)$ with uniform speed, which is \mathbb{Z}_2 -equivariant at any time of the deformation.

To show that $\|\mathbf{HDL}(G)\|$ is a strong \mathbb{Z}_2 -deformation retract of $\|\mathbf{DL}(G)\|$, we define simplicial complexes and simplicial \mathbb{Z}_2 -maps

$$\mathbf{DL}(G) =: S_0 \xrightarrow{f_0} S_1 \xrightarrow{f_1} \dots \xrightarrow{f_N} S_{N+1} := \mathbf{HDL}(G),$$

such that S_{i+1} is a \mathbb{Z}_2 -subcomplex of S_i and S_{i+1} is a strong \mathbb{Z}_2 -deformation retract of S_i . It turns out that the composition of the f_i yields the earlier defined map j , that is

$$j = f_N \circ \dots \circ f_1 \circ f_0.$$

To construct S_{i+1} inductively from S_i , we consider

$$X := \max_{\prec} \{Y \in J \mid Y \uplus \emptyset \in S_i\}$$

and obtain S_{i+1} from S_i by deleting each simplex of S_i that contains $X \uplus \emptyset$ or its \mathbb{Z}_2 -partner $\emptyset \uplus X$, i.e.,

$$S_{i+1} := \{\sigma \mid \sigma \in S_i \text{ and } X \uplus \emptyset \notin \sigma \text{ and } \emptyset \uplus X \notin \sigma\}.$$

The maximality of X implies that a maximal simplex which contains $X \uplus \emptyset$ (resp. $\emptyset \uplus X$) also contains $\emptyset \uplus \mathbf{CN}(X)$ (resp. $\mathbf{CN}(X) \uplus \emptyset$). Hence the map f_i defined on the vertices $v \in V(S_i)$ via

$$f_i(v) := \begin{cases} \emptyset \uplus \mathbf{CN}(X) & \text{if } v = X \uplus \emptyset \\ \mathbf{CN}(X) \uplus \emptyset & \text{if } v = \emptyset \uplus X \\ v & \text{otherwise} \end{cases}$$

is simplicial and \mathbb{Z}_2 -equivariant.

Thus $F: \|S_i\| \times [0, 1] \rightarrow \|S_i\|$ given by $F(x, t) := t \cdot x + (1 - t) \cdot \|f_i\|(x)$ is a well-defined \mathbb{Z}_2 -homotopy from $\|f_i\|$ to $\text{Id}_{\|S_i\|}$ that fixes $\|S_{i+1}\|$. \square

We end this section with a construction of a \mathbb{Z}_2 -map $\mathbf{HDL}(f)$ between $\mathbf{HDL}(G)$ and $\mathbf{HDL}(H)$ if we are given a graph homomorphism $f: G \rightarrow H$. Once we have chosen the partial orders that define the maps j_G and j_H that give $\mathbf{HDL}(G)$ and $\mathbf{HDL}(H)$, we simply compose the following simplicial \mathbb{Z}_2 -maps:

- The inclusion $\iota: \mathbf{HDL}(G) \rightarrow \mathbf{ssd}(\mathbf{B}(G))$,
- the map $\mathbf{ssd}(\mathbf{B}(f)): \mathbf{ssd}(\mathbf{B}(G)) \rightarrow \mathbf{ssd}(\mathbf{B}(H))$ canonically induced from f ,
- the map $\mathbf{cn}^2: \mathbf{ssd}(\mathbf{B}(H)) \rightarrow \mathbf{DL}(H)$, and
- the map $j_H: \mathbf{DL}(H) \rightarrow \mathbf{HDL}(H)$.

More precisely, the simplicial \mathbb{Z}_2 -map $\Psi : \text{HDL}(G) \rightarrow \text{HDL}(H)$ is defined by:

$$\Psi := j_H \circ \text{cn}^2 \circ \text{ssd}(\text{B}(f)) \circ \iota.$$

Since the halved doubled Lovász complex $\text{HDL}(G)$ is \mathbb{Z}_2 -isomorphic to the original Lovász complex $\text{L}(G)$, this map can be interpreted as a simplicial \mathbb{Z}_2 -map $\text{L}(f)$ between $\text{L}(G)$ and $\text{L}(H)$. This construction is significantly simpler than the construction of the \mathbb{Z}_2 -map $\text{L}(f) : \text{L}(G) \rightarrow \text{L}(H)$ described by Walker, [65].

3.4 THE $K_{\ell,m}$ -THEOREM

An upper bound of the lower bound for the chromatic number of graphs is provided under the assumption that a (not necessarily induced) complete bipartite subgraph of type $K_{\ell,m}$ does not exist.

Theorem 3.4.1. If a graph G does not contain a complete bipartite subgraph $K_{\ell,m}$, then the index of its box complex is bounded by

$$\text{ind}(\text{B}(G)) \leq \ell + m - 3.$$

We give two proofs for this theorem. The first one uses the shore subdivision and the halved doubled Lovász complex, the other is a direct argument on $\text{L}(G)$ along the lines of Walker [65]. Before we prove this theorem, we make two remarks. Firstly, this result is best possible, since $K_{\ell+m-1}$ does not contain a $K_{\ell,m}$ and $\text{ind}(\text{B}(K_{\ell+m-1})) = \ell + m - 3$; see [50, Lemma 5.9.2]. Secondly, this upper bound can become arbitrarily bad. Since $K_{1,k+1}$ is not a subgraph of $K_{k,k}$, we conclude from Theorem 3.4.1 that $\text{ind}(\text{B}(K_{k,k})) \leq k - 1$. But $\text{ind}(\text{B}(K_{k,k})) = 0$, since $K_{k,k}$ is bipartite.

Proof. (using shore subdivision) Let $\Phi : \text{ssd}(\text{B}(G)) \rightarrow \text{ssd}(\text{B}(G))$ be the simplicial \mathbb{Z}_2 -map defined by $j \circ \text{cn}^2$. Since the shore subdivision does not change the index, the index does not get smaller if we pass from one space to the image of a \mathbb{Z}_2 -map [50, Proposition 5.3.2], and the index is dominated by dimension [50, Proposition 5.3.2(v)], it suffices to show the last inequality of

$$\text{ind}(\text{B}(G)) = \text{ind}(\text{ssd}(\text{B}(G))) \leq \text{ind}(\text{Im } \Phi) \leq \dim(\text{Im } \Phi) \leq \ell + m - 3.$$

To estimate the dimension of $\text{Im } \Phi = \text{HDL}(G)$, we use that the graph G does not contain a subgraph of type $K_{\ell,m}$ and assume without loss of generality that $\ell \leq m$. A vertex of $\text{HDL}(G)$ or $\text{DL}(G)$ of the form $A \uplus \emptyset$ or $\emptyset \uplus A$ is called *small* if $|A| < \ell$, *medium* if $\ell \leq |A| < m$, and *large* if $m \leq |A|$. If $\ell = m$, medium vertices do not exist. Let $\sigma = \mathcal{A} \uplus \mathcal{B}$ be a simplex of $\text{HDL}(G)$ and consider the set of vertices

$$M_\sigma := V(j^{-1}(\sigma)) = \bigcup_{A \in \mathcal{A}} \{A \uplus \emptyset, \emptyset \uplus \text{CN}(A)\} \cup \bigcup_{B \in \mathcal{B}} \{\text{CN}(B) \uplus \emptyset, \emptyset \uplus B\}.$$

Clearly, $|M_\sigma|$ is at most twice $|V(\sigma)|$. If σ has a large vertex $A \uplus \emptyset$, then the vertex $\emptyset \uplus \text{CN}(A)$ must be small, otherwise G would contain a subgraph of type $K_{\ell,m}$. Hence there are at most $2 \cdot 2(\ell - 1)$ many vertices in M_σ that are large or small. Since the number of medium vertices is at most $2(m - \ell)$, we have

$$|M_\sigma| \leq 2 \cdot 2(\ell - 1) + 2(m - \ell) = 2(\ell + m - 2).$$

Hence $|V(\sigma)| \leq \ell + m - 2$ for all σ , and thus $\dim(\text{HDL}(G))$ is at most $\ell + m - 3$. \square

Proof. (using Lovász Complex) It suffices to prove $\dim(\mathbf{L}(G)) \leq \ell + m - 3$ since

$$\text{ind}(\mathbf{B}(G)) \leq \text{ind}(\mathbf{L}(G)) \leq \dim(\mathbf{L}(G)),$$

compare [51], or use that $\text{Im } \Phi = \text{HDL}(G) \simeq_{\mathbb{Z}_2} \mathbf{L}(G)$ by Section 3.3. Without loss of generality, let $\ell \leq m$ and $\sigma = A_1 \subset \dots \subset A_p$ be a simplex of $\mathbf{L}(G)$ of maximal dimension $p - 1$. If $p < \ell$ we are done. Thus suppose that $p \geq \ell$. Then $G[A_\ell; \text{CN}(A_\ell)]$ is a bipartite subgraph of G and we have $|A_\ell| \geq \ell$. Moreover, we have $|\text{CN}(A_\ell)| \geq p - \ell + 1$, since $|\text{CN}(A_j)| < |\text{CN}(A_\ell)|$ if $j > \ell$. The assumption that G does not contain a subgraph of type $K_{\ell,m}$ implies that $m > p - \ell + 1$, i.e., $\dim(\sigma) \leq \ell + m - 3$. \square

CHAPTER 4

GENERALISED KNESER COLOURINGS

INTRODUCTION

As indicated at the beginning of Part II, the topological method introduced by Lovász 1978 has been generalised in the subsequent years. In this chapter we are concerned with generalisations to hypergraphs. In the case of Kneser graphs and certain Kneser hypergraphs there is a combinatorial lower bound in terms of the colourability defect that is derived from generalisations of Lovász' topological lower bound.

The main point of this chapter is a careful distinction between different versions of r -uniform hypergraphs, where we assume always $r > 1$. We give precise definitions in Section 4.1, where we also define the colourability defect. The usual definition of an r -uniform hypergraph $H = (V(H), E(H))$ says that there is a node set $V(H) = [n]$ and a family $E(H)$ of subsets of $[n]$ of cardinality r , see Berge [11]. We call such a hypergraph an *r -uniform hypergraph without multiplicities*, since no hyperedge contains multiple copies of a node. We relax this definition a little bit if we speak of an *r -uniform hypergraph with multiplicities*, that is, we allow multiple copies of a node in a hyperedge. A hyperedge is a *loop* if it consists only of copies of one node. Hence r -uniform hypergraphs without multiplicities are always *loop-free* but r -uniform hypergraphs with multiplicities are either *loop-free* or not. Colouring an r -uniform hypergraph with loops is not really interesting since there will always occur monochromatic hyperedges. For technical reasons, we include hypergraphs with loops into our analysis. Moreover, we generalise the concept of a generalised Kneser graph to *r -uniform Kneser hypergraphs with and without multiplicities*. These two hypergraphs do not coincide in the setting studied by Sarkaria [56] and Ziegler [69]. Ziegler states his result as a colouring result of Kneser hypergraphs without multiplicities. We point out that the statement of Theorem 5.1 of Ziegler [69] does not hold in the generality claimed there. It is only valid for Kneser hypergraphs with multiplicities.

Section 4.2 is devoted to some examples and counterexamples. We analyse in detail an example to illustrate the two concepts of r -uniform Kneser hypergraphs and show by counterexample that neither Sarkaria's nor Ziegler's result holds if we replace the Kneser hypergraph with multiplicities by a Kneser hypergraph without multiplicities.

In Section 4.4 we extend a result from Alon, Frankl, and Lovász [3] for multiplicity-free r -uniform hypergraphs if r is prime and from Özaydin [54] if r is a prime-power: We give a topological lower bound of the chromatic number of an r -uniform hypergraph with or without multiplicities if r is the power of a prime. The proof in this generality uses a result of Volovikov [64] for fixed-point free group actions. A summary of definitions and facts

needed is given in Section 4.3.

In Section 4.5 we give a new proof of Ziegler's result for r -uniform s -disjoint Kneser hypergraphs with multiplicities. The main tool used is *Sarkaria's inequality*. The proof is inspired by Matoušek's proof [49] of the result of Kríž [40, 41]. In fact we prove a slightly modified version of Ziegler's result: Firstly, Ziegler allows varying multiplicity s for the elements of the ground set considered. This is essential for his *combinatorial* proof. We restrict to the simplified case of constant s . A refined analysis of the deleted join should be possible to extend the presented proof to Ziegler's generality. Secondly, we prove a little bit more than Ziegler: The colourability defect of a set system \mathcal{T} is not the only lower bound for the chromatic number of the r -uniform Kneser hypergraph associated to \mathcal{T} . If r is prime we insert the index of an associated simplicial complex between these two numbers. This topological lower bound might yield better estimates of the chromatic number, but is hard to compute in general. The proofs for r -uniform Kneser hypergraphs usually consist of two parts. Firstly, one proves the case that r is a prime. Then a *combinatorial argument* that traces back to Alon, Frankl, and Lovász[3] is used to prove the remaining cases by induction. As pointed out by Vogel [63], the induction presented by Ziegler is not complete. We give a partial solution of the problem that is a bit simpler compared to the one by Lange and Vogel described in [63].

For arbitrary r -uniform hypergraphs with or without multiplicities, the known methods can only be used to prove topological lower bounds for the chromatic number as stated in Theorem 4.4.1 if r is a prime-power. If one tries to prove the topological lower bound for arbitrary r , one faces the same difficulties that are faced by any proof of the topological Tverberg conjecture. In case of graphs the result for Kneser graphs translates to general graphs since every graph G is a generalised Kneser graph, [51], that is, there is a system of subsets \mathcal{T} of a some ground set $[n]$ such that $\text{KG}_1^2\mathcal{T} = G$. Such a result for r -uniform hypergraphs and arbitrary r would imply a lower bound for all r -uniform hypergraph. But for r -uniform hypergraphs (multiplicity-free or not) it is not even known whether a realisation as an r -uniform s -disjoint Kneser hypergraph for some s and an appropriate set system exists or not.

4.1 PRELIMINARIES

In this section we introduce the fundamental objects studied in this chapter. Firstly, the s -disjoint r -colourability defect is defined in a version due to Ziegler [69], which generalises a concept of Dol'nikov [24]. The colourability defect serves as a *combinatorial* lower bound for the chromatic number of certain uniform Kneser hypergraphs. Secondly, we define r -uniform hypergraphs with and without multiplicities. The latter version is the usual notion of a uniform hypergraph as described by Berge [11], the other one is a relaxation of this concept to hypergraphs that allows *semi-loops* and *loops*, that is, a hyperedge is allowed to contain multiple copies of a node. We then specialise this concept to uniform Kneser hypergraphs. Sarkaria [56] formulated his result not for hypergraphs, but for set systems. This is easily translated to the uniform Kneser hypergraph terminology *with multiplicities*.

s-disjoint sets. For integers $r, s \geq 1$ we say that subsets S_1, \dots, S_r of $[n]$ are s -disjoint if each element of $[n]$ occurs in at most s of the sets S_i , or equivalently, if the intersection of any choice of $s + 1$ sets is empty. The latter formulation is the reason that this concept is called $(s + 1)$ -wise disjoint by Sarkaria [56]. We emphasise that $S_i = S_j$ may occur for $i \neq j$. Obviously, for $r \leq s$ there is no restriction on the $S_j \subseteq [n]$.

s-disjoint r-colourability defect. The collection of elements of $[n]$ where each element occurs with multiplicity s is the *multiset* $[n]^s$. The s -disjoint r -colourability defect $\text{cd}_s^r \mathcal{S}$ of a set $\mathcal{S} \subseteq 2^{[n]}$ is the minimal number of elements one has to remove from the multiset $[n]^s$ such that the remaining multiset can be covered by r subsets of $[n]$ such that none of the sets contains an element from \mathcal{S} . These sets may have non-empty intersection and multiple copies of a set may occur. This number can be computed by evaluating

$$\text{cd}_s^r \mathcal{S} = n \cdot s - \max \left\{ \sum_{j=1}^r |R_j| \mid \begin{array}{l} R_1, \dots, R_r \subseteq [n] \text{ } s\text{-disjoint} \\ \text{and } S \not\subseteq R_j \text{ for all } S \in \mathcal{S} \text{ and all } j \end{array} \right\}.$$

Obviously, $\text{cd}_s^r \emptyset = n(s - r)$ if $r \leq s$ since we can only cover r copies of $[n]$ of $[n]^s$.

r-multisubsets of $[n]$. A collection x_1, \dots, x_r of elements of $[n]$ is called an r -multisubset of $[n]$. We denote an r -multiset by $\{x_1, \dots, x_r\}$.

r-uniform hypergraphs with or without multiplicities. Consider $\mathcal{S} \subseteq 2^{[n]}$ such that $\bigcup_{S \in \mathcal{S}} S = [n]$ and every $S \in \mathcal{S}$ has cardinality r . The node set $V(H)$ of the r -uniform hypergraph $H = (V(H), E(H))$ without multiplicities is $[n]$ and the hyperedges $E(H)$ are \mathcal{S} . We often refer to the hypergraph H by its hyperedge set \mathcal{S} . An r -uniform hypergraph without multiplicities coincides with Berge's definition of an r -uniform hypergraph, [11].

Let \mathcal{S}' be a set of r -multisubsets of $[n]$. The r -uniform hypergraph $H' = (V(H'), E(H'))$ with multiplicities has node set $V(H') = [n]$ and hyperedge set $E(H') = \mathcal{S}'$. Again, we often refer to H' simply by its hyperedge set \mathcal{S}' .

A hypergraph contains a *loop* if there is a hyperedge that contains copies of only one node. We note that an r -uniform hypergraph without multiplicities coincides with Berge's definition of an r -uniform hypergraph, [11].

r-uniform s-disjoint Kneser hypergraphs. For a set $\mathcal{T} = \{T_1, \dots, T_m\}$ of subsets of $[n]$, we consider the r -uniform s -disjoint Kneser hypergraph $\text{KG}_s^r \mathcal{T}$ with multiplicities

on the node set $V(KG_s^r \mathcal{T}) = [m]$ with hyperedges

$$E(KG_s^r \mathcal{T}) := \left\{ \{k_1, \dots, k_r\} \mid \begin{array}{l} \{k_1, \dots, k_r\} \text{ is an } r\text{-multisubset of } [m], \\ \text{and } T_{k_1}, \dots, T_{k_r} \text{ are } s\text{-disjoint} \end{array} \right\}.$$

If $r > s$ then $KG_s^r \mathcal{T}$ is a *loop-free* r -uniform s -disjoint Kneser hypergraph with multiplicities. It does contain loops if $r \leq s$.

The r -uniform s -disjoint Kneser hypergraph $kg_s^r \mathcal{T}$ without multiplicities has the same node set $V(kg_s^r \mathcal{T}) = [m]$ and the following hyperedge set:

$$E(kg_s^r \mathcal{T}) := \left\{ \{k_1, \dots, k_r\} \mid \begin{array}{l} \{k_1, \dots, k_r\} \text{ is an } r\text{-subset of } [m] \\ \text{and } T_{k_1}, \dots, T_{k_r} \text{ are } s\text{-disjoint} \end{array} \right\}.$$

The r -uniform s -disjoint Kneser hypergraph without multiplicities is *loop-free* even for $r \leq s$. We can obtain $kg_s^r \mathcal{T}$ from $KG_s^r \mathcal{T}$ by discarding hyperedges. In this sense, $kg_s^r \mathcal{T}$ is a subhypergraph of $KG_s^r \mathcal{T}$. In the special case $s = 1$ we have $KG_s^r \mathcal{T} = kg_s^r \mathcal{T}$ since an r -multiset with r pairwise disjoint elements can be seen as an r -set. In particular for $r = 2$ and $s = 1$, both definitions specialise to a Kneser graph of $\mathcal{T} \subseteq 2^{[n]}$.

Colourings. There are different concepts to colour hypergraphs that extend the notion from graph theory. The one we are interested in was introduced by Erdős and Hajnal in 1966, [25]. A colouring of an r -uniform hypergraph \mathcal{S} (multiplicity-free or not) with k colours is a mapping $c : V(\mathcal{S}) \rightarrow [k]$ that assigns to each node of \mathcal{S} a colour so that no hyperedge is monochromatic, that is, for each $e \in E(\mathcal{S})$ we have $|\{c(x) \mid x \in e\}| \geq 2$. The *chromatic number* $\chi(\mathcal{S})$ is the smallest number m such that there is a colouring of \mathcal{S} with m colours. Every hyperedge of $kg_s^r \mathcal{T}$ is a hyperedge of $KG_s^r \mathcal{S}$, hence:

$$\chi(kg_s^r \mathcal{T}) \leq \chi(KG_s^r \mathcal{T}).$$

Unfortunately, we have to deal with a number of degenerate cases. Since $KG_s^r \mathcal{T}$ contains loops for $r \leq s$, we define $\chi(KG_s^r \mathcal{T}) = \infty$ in this case. If $\mathcal{T} = \emptyset$, there are no vertices to colour, so $\chi(KG_s^r \emptyset) = 0$.

Generalising Lovász' result. We now state Lovász' result and its generalisations to Kneser hypergraphs in a unified language. We emphasise that neither Lovász, nor Alon, Frankl, and Lovász, nor Kříž, nor Sarkaria use the concept of the colourability defect which is due to Dol'nikov. Instead, the original articles give explicit lower bounds which equal the corresponding colourability defect stated below. The translation is straight-forward; for example in case of Sarkaria [56] we proceed as follows:

“If $N(j-1) - 1 \geq M(p-1) + p(S-1)$, then any colouring of the S -subsets of an N -set by M colours must yield a p -tuple of S -subsets having the same colour *and* such that the intersection of any j of the sets is empty.”

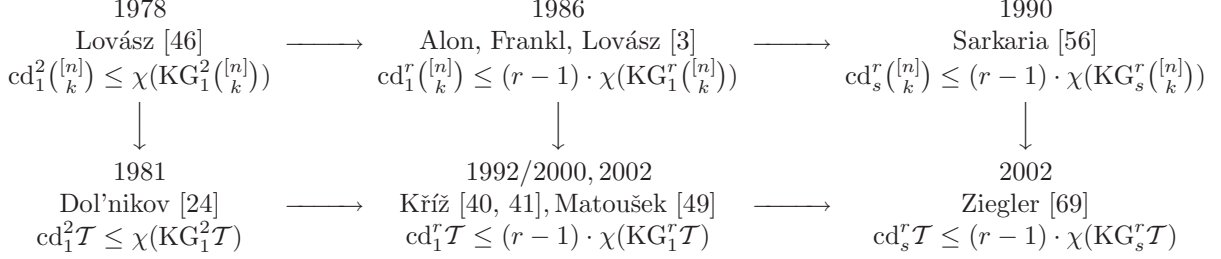
So, in our language, he colours the p -uniform $(j-1)$ -disjoint Kneser hypergraph $KG_{j-1}^p \binom{[N]}{S}$ with multiplicities and shows that a colouring with M colours must yield a monochromatic hyperedge if $N(j-1) - 1 \geq M(p-1) + p(S-1)$. We know from Ziegler [69, Lemma 3.2]

$$\text{cd}_{j-1}^p \binom{[N]}{S} = \max\{N(j-1) - p(S-1), 0\},$$

we obtain

$$(p-1)\chi(\text{KG}_{j-1}^p([N]_S)) \geq N(j-1) - p(S-1) = \text{cd}_{j-1}^p([N]_S).$$

The following diagram relates results of articles published between 1978 and 2002 in the unified language. The statement “ A is generalised by B ” is indicated by $A \longrightarrow B$.



A remark on Theorem 5.1 of [69]. The result by Ziegler is more general than stated here. It still holds, if we allow different multiplicities for the elements of the ground set, that is, if we consider a *vector* \mathbf{s} . We restrict to the case of where s_i is constant for $1 \leq i \leq n$. Moreover, Ziegler states his Theorem 5.1 for $\text{kg}_s^r\mathcal{T}$, that is, for r -uniform s -disjoint Kneser hypergraphs in the sense of Berge. The proof on page 679 of [69] yields the desired contradiction only if one assumes a colouring of $\text{KG}_s^r\mathcal{T}$, that is, for r -uniform s -disjoint Kneser hypergraphs with multiplicities. More precisely, the construction only guarantees that the p subsets S_1, \dots, S_p of $[n]$ are s -disjoint, they need not be pairwise different. We emphasise that this part of the proof is correct if we consider $\text{KG}_s^r\mathcal{T}$ instead of $\text{kg}_s^r\mathcal{T}$. In the following section we give examples that show that the statement for $\text{kg}_s^r\mathcal{T}$ is not true in general. Moreover, Ziegler claims that the theorem is true for all integers r . Unfortunately, there is a gap in the argument that derives the general case from the case that r is prime. We discuss the problem in detail in the last section, when we give an alternative proof that generalises an Matoušek’s proof of the result of Kříž.

4.2 EXAMPLES AND COUNTEREXAMPLES

We illustrate the two concepts of r -uniform s -disjoint Kneser hypergraphs by example. Furthermore, we give examples which show that Sarkaria’s and Ziegler’s results do not hold for r -uniform s -disjoint Kneser hypergraphs without multiplicities.

To define an r -uniform s -disjoint Kneser hypergraph, it was convenient to use the index set of \mathcal{T} to define the hyperedges. In the examples, it is more convenient to identify an hyperedge with a collection of nodes.

Example 1. We give an example to illustrate the two concepts of s -disjoint r -uniform Kneser hypergraphs and to see that the chromatic numbers $\chi(\text{kg}_s^r\mathcal{S})$ and $\chi(\text{KG}_s^r\mathcal{S})$ can be different. We restrict ourselves to the following small but interesting case: $r = 3$, $s = 2$, and $\mathcal{T} \subseteq \binom{[5]}{2}$. Let us consider

$$\mathcal{T} := \{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\} \}.$$

The 3-uniform 2-disjoint Kneser hypergraph $\text{kg}_2^3\mathcal{T}$ without multiplicities has 6 nodes (the elements of \mathcal{T}). Let $x \in \{2, 3, 4, 5\}$. Any hyperedge consists either of two nodes of type $\{1, x\}$ plus $\{2, 3\}$ or $\{4, 5\}$, or it consists of $\{2, 3\}$, $\{4, 5\}$ plus one node of type $\{1, x\}$. Explicitly, we have the hyperedges

$$\begin{array}{lll} \{ \{1, 2\}, \{1, 3\}, \{2, 3\} \} & \{ \{1, 2\}, \{1, 3\}, \{4, 5\} \} & \{ \{1, 2\}, \{2, 3\}, \{4, 5\} \} \\ \{ \{1, 2\}, \{1, 4\}, \{2, 3\} \} & \{ \{1, 2\}, \{1, 4\}, \{4, 5\} \} & \{ \{1, 3\}, \{2, 3\}, \{4, 5\} \} \\ \{ \{1, 2\}, \{1, 5\}, \{2, 3\} \} & \{ \{1, 2\}, \{1, 5\}, \{4, 5\} \} & \{ \{1, 4\}, \{2, 3\}, \{4, 5\} \} \\ \{ \{1, 3\}, \{1, 4\}, \{2, 3\} \} & \{ \{1, 3\}, \{1, 4\}, \{4, 5\} \} & \{ \{1, 5\}, \{2, 3\}, \{4, 5\} \} \\ \{ \{1, 3\}, \{1, 5\}, \{2, 3\} \} & \{ \{1, 3\}, \{1, 5\}, \{4, 5\} \} & \\ \{ \{1, 4\}, \{1, 5\}, \{2, 3\} \} & \{ \{1, 4\}, \{1, 5\}, \{4, 5\} \} & \end{array}$$

Colouring $\text{kg}_2^3\mathcal{T}$ means therefore colouring the edges of the graph \mathcal{T} , shown in Figure 4.1, such that no 2-disjoint triple of edges that form a hyperedge is monochromatic. This can be done with 2 colours as indicated in Figure 4.1: Colour each node of $\text{kg}_2^3\mathcal{T}$ that contains 1 with one colour, and colour $\{2, 3\}$ and $\{4, 5\}$ with the other colour.

The s -disjoint r -uniform Kneser hypergraph $\text{KG}_2^3\mathcal{T}$ with multiplicities has the following additional hyperedges among many other that we do not list explicitly:

$$\{ \{ \{2, 3\}, \{2, 3\}, \{4, 5\} \} \} \quad \text{and} \quad \{ \{ \{2, 3\}, \{4, 5\}, \{4, 5\} \} \}.$$

These hyperedges force us to use at least three colours, that is, $\chi(\text{KG}_2^3\mathcal{T})$; for a colouring see Figure 4.2. The sets $R_1 = \{2, 4\}$, $R_2 = \{2, 5\}$ and $R_3 = \{3, 4\}$ are 2-disjoint, i.e., $\text{cd}_s^r\mathcal{T} \leq 4$. It follows from Theorem 5.1 of Ziegler [69] or Theorem 4.5.1 that $\frac{\text{cd}_2^3\mathcal{T}}{3-1} \leq \chi(\text{KG}_2^3\mathcal{T})$. In this particular example we have

$$\frac{\text{cd}_2^3\mathcal{T}}{3-1} \leq 2 = \chi(\text{kg}_2^3\mathcal{T}) < \chi(\text{KG}_2^3\mathcal{T}) = 3.$$

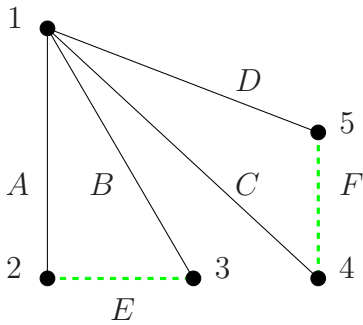


Figure 4.1: The graph \mathcal{T} . Its edges A, B, C, D, E , and F represent the nodes of $\text{kg}_s^r\mathcal{T}$. They are coloured such that the hypergraph $\text{kg}_s^r\mathcal{T}$ without multiplicities is coloured properly.

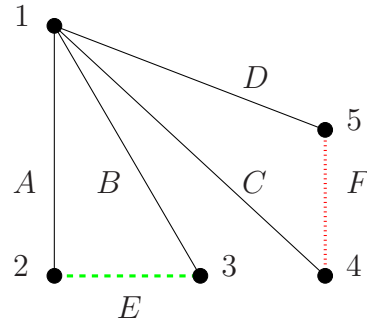


Figure 4.2: Again, the edges A, B, C, D, E , and F of the graph \mathcal{T} represent the nodes of $\text{KG}_s^r\mathcal{T}$. They are coloured such that the hypergraph $\text{KG}_s^r\mathcal{T}$ with multiplicities is coloured properly.

This example shows that s -disjoint r -uniform Kneser hypergraphs with and without multiplicities have different chromatic numbers in general. To show that Sarkaria's and Ziegler's result does not hold for s -disjoint r -uniform Kneser hypergraphs *without* multiplicities, we have to work a little bit more. The next (counter-)example looks at first sight like a straight-forward generalisation of Example 1 to an n -element ground set. But this is not true: Example 1 is concerned with r -uniform Kneser hypergraphs of type KG_{r-1}^r and kg_{r-1}^r while the next paragraph studies the type kg_{r-2}^r .

Counterexample 1. We now consider an example similar to Example 1 to show that the colourability defect is not a lower bound for $\text{kg}_{r-2}^r \mathcal{T}$ in general. For fixed $n \geq 5$, we consider the following set \mathcal{T} of subsets of $[n]$:

$$\mathcal{T} := \{ \{1, 2\}, \dots, \{1, n\}, \{2, 3\}, \{4, 5\} \}.$$

The $(r-2)$ -disjoint r -uniform Kneser hypergraph $\text{kg}_{r-2}^r \mathcal{T}$ is easily described. Every hyperedge contains $r-2$ different elements of $\{ \{1, 2\}, \dots, \{1, n\} \}$ plus $\{2, 3\}$ and $\{4, 5\}$. Therefore, $\chi(\text{kg}_{r-2}^r \mathcal{T}) = 2$ if $n \geq r-1$. We now want to compute the $(r-2)$ -disjoint r -colourability defect of \mathcal{T} . We have to cover the multiset $[n]^{r-2}$ (each element of $[n]$ has multiplicity $r-2$) by r sets R_1, \dots, R_r so that no set contains an element of \mathcal{T} . Obviously, $R_t = \{1\}$ if $1 \in R_t$. Let r_1 denote the number of sets R_t that contain 1. The sets R_u that do not contain 1 cannot contain 2 and 3 at the same time. Similarly R_u does not contain 4 and 5 at the same time. There are $r_2 = r - r_1$ such sets R_u . Therefore, we have not covered

$$\begin{aligned} & (r-2) - r_1 \text{ copies of } 1, \\ & 2(r-2) - r_2 \text{ copies of } 2 \text{ or } 3, \text{ and} \\ & 2(r-2) - r_2 \text{ copies of } 4 \text{ or } 5. \end{aligned}$$

In other words, at least

$$(r-2) - r_1 + 2(r-2) - r_2 + 2(r-2) - r_2 = 3r - 10 + r_1 \geq 3r - 10$$

elements are not covered. Hence, $\text{cd}_{r-2}^r \mathcal{T} \geq 3r - 10$. For $r > 8$ this implies

$$\text{cd}_{r-2}^r \mathcal{T} > (r-1)\chi(\text{kg}_{r-2}^r \mathcal{T}).$$

Thus we have shown that Ziegler's Theorem 5.1 does not hold for r -uniform s -disjoint Kneser hypergraphs $\text{kg}_{r-2}^r \mathcal{T}$ without multiplicities as just described.

Counterexample 2. We now show that the colourability defect is not a lower bound for $\text{kg}_{r-1}^r \binom{[n]}{2}$ in general, that is, Sarkaria's result is not true for Kneser hypergraphs without multiplicities for the parameters $s = r-1$ and $k = 2$. From Ziegler [69, Lemma 3.2], we know that

$$\text{cd}_s^r \binom{[n]}{2} = \max\{ns - r(k-1), 0\} = \max\{n(r-1) - r, 0\}.$$

Hence we have to show that

$$(r-1)\chi(\text{kg}_{r-1}^r \binom{[n]}{2}) < n(r-1) - r.$$

It suffices to colour $\text{kg}_{r-1}^r\binom{[n]}{2}$ with $n - 2$ colours. This can be done by a greedy-type colouring that was probably already known to Kneser in the case of graphs: Assign colour i to $T \in \mathcal{T}$ if i is the smallest element of T and $i \leq n - 3$. The elements not coloured yet are $\{n - 2, n - 1\}$, $\{n - 2, n\}$, and $\{n - 1, n\}$; too few to form a hyperedge. We colour these elements by colour $n - 2$. This suffices as a counterexample to Sarkaria's theorem for r -uniform s -disjoint Kneser hypergraphs $\text{kg}_{r-1}^r\binom{[n]}{2}$ without multiplicities.

4.3 GROUPS ACTING ON SIMPLICIAL COMPLEXES

This section summarises standard definitions and known facts on free and fixed-point free group actions. The complexes P_s^r and $P_s^r\mathcal{S}$ are needed in Section 4.5. A more detailed treatment can be found in Matoušek's textbook [50].

Deleted Joins. We generalise the notion introduced in Section 3.1 of Chapter 3. For sets A_1, \dots, A_t , we define

$$A_1 \uplus \dots \uplus A_t := \{(a, 1) \mid a \in A_1\} \cup \dots \cup \{(a, t) \mid a \in A_t\}.$$

For a simplicial complex K and positive integers $r \geq s$ the r -fold s -wise deleted join $K_{\Delta(s)}^{*r}$ is defined as

$$K_{\Delta(s)}^{*r} := \{F_1 \uplus \dots \uplus F_r \mid F_i \in K \text{ and } F_1, \dots, F_r \text{ is } s\text{-wise disjoint}\}.$$

To avoid confusion, we emphasise that in the definition s indicates s -wise disjointness, not s -disjointness, that is, we force the intersection of any $s + 1$ of the F_i to be empty.

Free \mathbb{Z}_r -spaces and \mathbb{Z}_r -index. A free \mathbb{Z}_r -space is a topological space X together with a free \mathbb{Z}_r -action Φ , i.e., for all $g, h \in \mathbb{Z}_r$ we have $\Phi(g) \circ \Phi(h) = \Phi(g + h)$, $\Phi(0) = \text{Id}$, and $\Phi(g)$ has no fixed point for $g \in \mathbb{Z}_r \setminus \{0\}$. A continuous map f between \mathbb{Z}_r -spaces (X, Φ_X) and (Y, Φ_Y) is \mathbb{Z}_r -equivariant (or a \mathbb{Z}_r -map for simplicity) if f commutes with the \mathbb{Z}_r -actions, i.e., $f \circ \Phi_X = \Phi_Y \circ f$. A simplicial complex (K, Φ) is a free simplicial \mathbb{Z}_r -space (or a free simplicial \mathbb{Z}_r -complex) if $\Phi : K \rightarrow K$ is a simplicial map such that $\|\Phi\|$ is a free \mathbb{Z}_r -action on $\|K\|$. A simplicial \mathbb{Z}_r -equivariant map f is a simplicial map between two simplicial \mathbb{Z}_r -spaces that commutes with the free \mathbb{Z}_r -actions. An important class of free \mathbb{Z}_r -spaces is formed by $E_n\mathbb{Z}_r$ -spaces: A free \mathbb{Z}_r -space is an $E_n\mathbb{Z}_r$ -space if it is n -dimensional and $(n - 1)$ -connected. The most prominent example for an $E_n\mathbb{Z}_r$ -space is the $(n + 1)$ -fold join $(\mathbb{Z}_r)^{*(n+1)}$ of an r -element set \mathbb{Z}_r .

The \mathbb{Z}_r -index $\text{ind}_{\mathbb{Z}_r}(X)$ of (X, Φ) is the smallest n such that a \mathbb{Z}_r -map from X to some $E_n\mathbb{Z}_r$ -space exists. A generalised Borsuk–Ulam theorem, e.g., Dold's theorem [23] for free group actions, provides the index for $E_n\mathbb{Z}_r$ -spaces: There is no \mathbb{Z}_r -map from $E_n\mathbb{Z}_r$ to $E_{n-1}\mathbb{Z}_r$. Hence $\text{ind}_{\mathbb{Z}_r}(E_n\mathbb{Z}_r) = n$. Since we consider cyclic shifts as group actions most of the time, we tend to refer to a \mathbb{Z}_r -space X without explicit reference to Φ .

Examples: P_s^r and $P_s^r\mathcal{S}$. Let $1 \leq s < r$ where r is a prime. Consider the poset P_s^r of s -disjoint r -tuples (S_1, \dots, S_r) of subsets of $[n]$ with $\bigcup_{i \in [r]} S_i \neq \emptyset$ ordered by componentwise inclusion, that is, $(S_1, \dots, S_r) \leq (T_1, \dots, T_r)$ if $S_i \subseteq T_i$ for all $1 \leq i \leq r$. The order complex

of a poset is the abstract simplicial complex that has the elements of the poset as vertices and simplices are formed by the chains of the poset. The barycentric subdivision of a simplicial complex X is the order complex of the face lattice of X . To simplify the notation in Section 4.5, we denote the order complex of P_s^r by P_s^r . The simplicial complex P_s^r can be interpreted as the barycentric subdivision of the r -fold $(s+1)$ -wise deleted join of an $(n-1)$ -simplex σ^{n-1} , which is homotopy equivalent to a wedge of $(ns-1)$ -dimensional spheres, see Matoušek [50] or alternatively Sarkaria [56]. Hence P_s^r is an $E_{ns-1}\mathbb{Z}_r$ -space. Another space we shall need in Section 4.5 is the order complex $P_s^r\mathcal{S}$ of the subposet $P_s^r\mathcal{S}$ of P_s^r that consists only of those r -tuples (S_1, \dots, S_r) that satisfy $\sum_{i=1}^r |S_i| \geq ns - \text{cd}_s^r\mathcal{S} + 1$, where $\mathcal{S} \subseteq 2^{[n]}$. By the definition of the s -disjoint r -colourability defect, such tuples have the useful property that there is an $S \in \mathcal{S}$ such that $S \subseteq S_i$ for some i .

Sarkaria's inequality. A useful inequality concerning the \mathbb{Z}_r -index of the join $K * L$ of two free simplicial \mathbb{Z}_r -complexes K and L is *Sarkaria's inequality*, see Matoušek [49, 50]:

$$\text{ind}_{\mathbb{Z}_r}(K * L) \leq \text{ind}_{\mathbb{Z}_r}(K) + \text{ind}_{\mathbb{Z}_r}(L) + 1.$$

By definition of the index, we have \mathbb{Z}_r -maps $K \longrightarrow (\mathbb{Z}_r)^{*(\text{ind}_{\mathbb{Z}_r}(K)+1)}$ and $L \longrightarrow (\mathbb{Z}_r)^{*(\text{ind}_{\mathbb{Z}_r}(L))}$ that induce a \mathbb{Z}_r -map

$$K * L \longrightarrow (\mathbb{Z}_r)^{*(\text{ind}_{\mathbb{Z}_r}(K)+1+\text{ind}_{\mathbb{Z}_r}(L)+1)}.$$

We shall need this inequality in Section 4.5. As historical aside, we note that Živaljević was the first who isolated *Sarkaria's inequality* in [70] for \mathbb{Z}_2 -actions, although the ideas can be implicitly found for example in Sarkaria [56].

Fixed-point free actions. A *fixed-point free action* Φ of a group G of order r on a topological space X means that no $x \in X$ is fixed by all $g \in G$. Obviously, a free group action is also fixed-point free. A standard example is the case $r = p^t$ a prime-power and $G = (\mathbb{Z}_p)^t = \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$ acting on $(\mathbb{R}^m)^r$ with the diagonal $\{(v, \dots, v) \mid v \in \mathbb{R}^m\}$ removed: Order the elements of $(\mathbb{Z}_p)^t$ and interpret $(\mathbb{Z}_p)^t$ as a subgroup of the symmetric group of degree r by the obvious $(\mathbb{Z}_p)^t$ -action on itself. We now have a standard action of G on $(\mathbb{R}^m)^r$ by permuting the r copies of \mathbb{R}^m . The action is fixed-point free on $(\mathbb{R}^m)^r \setminus \{(v, \dots, v) \mid v \in \mathbb{R}^m\}$ for all r . It is free if r is a prime. The space $(\mathbb{R}^m)^r \setminus \{(v, \dots, v) \mid v \in \mathbb{R}^m\}$ is homotopy equivalent to an $(m(r-1)-1)$ -sphere.

Theorem 4.3.1 (“Volovikov’s theorem”, [64]). Let $r = p^t$ be a power of a prime and consider a fixed-point free action of $(\mathbb{Z}_p)^t = \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$ on X and Y . Suppose that for all $i \leq \ell$ we have $\tilde{H}^i(X; \mathbb{Z}_p) = 0$ and that Y is a finite-dimensional cohomology k -dimensional sphere over the field \mathbb{Z}_p . If there exists a $(\mathbb{Z}_r)^t$ -equivariant map $f : X \longrightarrow Y$, then $\ell < k$.

Box complexes. Alon, Frankl, and Lovász [3], Kříž [40], and Matoušek and Ziegler [51] describe different versions of a box complex to obtain topological lower bounds for the chromatic number of a graph or r -uniform hypergraph (without multiplicities). We now define a box complex $B_0(\mathcal{S})$ associated to an r -uniform hypergraph \mathcal{S} with or without multiplicities which in the graph case reduces to the box complexes $B_0(G)$ described by Matoušek and Ziegler [51], but not to $B(G)$, as indicated at the end of this paragraph. In

particular, the box complex $B_0(\mathcal{S})$ differs from $B(\mathcal{S})$ defined and used in the preceeding chapter. This complex is different from (but closely related to) the complexes considered by Alon, Frankl, and Lovász [3] and Kříž [40]. For an r -uniform hypergraph \mathcal{S} without multiplicities we define

$$B_0(\mathcal{S}) := \left\{ (U_1, \dots, U_r) \mid \begin{array}{l} U_i \subseteq V(\mathcal{S}), \bigcup_{i \in [r]} U_i \neq \emptyset, \text{ and if all } U_i \neq \emptyset, \text{ then} \\ u_i \in U_i \ (1 \leq i \leq r) \text{ implies } \{u_1, \dots, u_r\} \in E(\mathcal{S}) \end{array} \right\}.$$

In case of an r -uniform hypergraph \mathcal{S}' with multiplicities we replace the r -set $\{u_1, \dots, u_r\} \in E(\mathcal{S})$ in the definition by the r -multisubset $\{\{u_1, \dots, u_r\}\} \in E(\mathcal{S}')$. The box complex of an r -uniform hypergraph has a free action by cyclic shifting if r is prime and a fixed-point free action for arbitrary r . Since $\text{kg}_s^r \mathcal{S}$ (considered as a hypergraph with multiplicities) is a subhypergraph of $\text{KG}_s^r \mathcal{S}$ we have for prime r :

$$\text{ind}_{\mathbb{Z}_r}(B_0(\text{kg}_s^r \mathcal{S})) \leq \text{ind}_{\mathbb{Z}_r}(B_0(\text{KG}_s^r \mathcal{S})).$$

For a graph G , the complexes $B(G)$ and $B_0(G)$ differ slightly: The neighbourhood of the nodes that correspond to one shore is empty by definition, so the vertex set of a shore never forms a simplex in $B(G)$, but they do in $B_0(G)$.

Colour complexes. The r -uniform colour complex \mathcal{C}_c is the simplicial complex $(\sigma^{c-1})_{\Delta(r)}^{*r}$, i.e., the vertex set $V(\mathcal{C})$ consists of r copies of $[c]$ and the simplices are ordered r -tuples (C_1, \dots, C_r) of subsets of $[c]$ with $\bigcup_{i \in [r]} C_i \neq \emptyset$ and $\bigcap_{j=1}^r C_j = \emptyset$. The cyclic group \mathbb{Z}_r acts on the colour complex by cyclic shift of the components. This action is free if r is prime.

4.4 A TOPOLOGICAL LOWER BOUND FOR THE CHROMATIC NUMBER OF HYPERGRAPHS

Theorem 4.4.1. Let $r = p^t$ for a prime p and a positive integer t . Consider an r -uniform hypergraph \mathcal{S} with or without multiplicities, but without loops. Suppose there is an ℓ such that $H^i(B_0(\mathcal{S}); \mathbb{Z}_p) = 0$ for $i \leq \ell$. Then

$$\chi(\mathcal{S}) \geq \left\lceil \frac{\ell + 2}{r - 1} \right\rceil.$$

The content of Theorem 4.4.1 is certainly well-known to the experts in case of hypergraphs *without* multiplicities and probably no surprise in case of hypergraphs *with* multiplicities. But the only reference for hypergraphs without multiplicities known to the author is an unpublished manuscript by Özaydin [54]. For that reason we supply a proof. Volovikov [64] uses Theorem 4.3.1 to generalise the topological Tverberg theorem to prime-powers. So far it is not possible to generalise Theorem 4.4.1 or the topological Tverberg theorem from a prime-power r to general r .

Proof. Any colouring $c : V(\mathcal{S}) \rightarrow [\chi(\mathcal{S})]$ induces a continuous \mathbb{Z}_r -map

$$f_c : \|B_0(\mathcal{S})\| \rightarrow (\mathbb{R}^{\chi(\mathcal{S})})^r \setminus \{(v, \dots, v) \mid v \in \mathbb{R}^{\chi(\mathcal{S})}\}$$

which we define as follows. Consider the standard basis $e_1, \dots, e_{r \cdot \chi(\mathcal{S})}$. We now map a vertex $(\emptyset, \dots, \emptyset, v, \emptyset, \dots, \emptyset)$ that has non-empty coordinate j to $e_{j+r(c(v)-1)}$ and extend affinely. The image of this map is contained in the boundary of the simplex that is given by the convex hull of $e_1, \dots, e_{r \cdot \chi(\mathcal{S})}$. Moreover, the image does not meet $\{(v, \dots, v) \mid v \in \mathbb{R}^{\chi(\mathcal{S})}\}$ since c is a colouring of the hypergraph. In particular, a $(\mathbb{Z}_p)^t$ -homotopic copy of $\text{Im } f_c$ is contained in a sphere of dimension $((r-1) \cdot \chi(\mathcal{S}) - 1)$ by normalising each point of $\text{Im } f_c$. The spaces $B_0(\mathcal{S})$ and $\text{Im } f_c$ (as well as its homotopic copy) are fixed-point free $(\mathbb{Z}_p)^t$ -spaces, hence we can apply Volovikov's theorem (Theorem 4.3.1) to deduce $\ell < (r-1) \cdot \chi(\mathcal{S}) - 1$. \square

To prove the combinatorial lower bound of Theorem 4.5.1 for the chromatic number of r -uniform s -disjoint Kneser hypergraphs with multiplicities, the following weaker statement for prime numbers suffices. We include its proof since we can avoid Volovikov's theorem.

Theorem 4.4.2. Let r be a prime and \mathcal{S} be an r -uniform hypergraph with or without multiplicities. Then

$$\chi(\mathcal{S}) \geq \left\lceil \frac{\text{ind}_{\mathbb{Z}_r}(B_0(\mathcal{S})) + 1}{r - 1} \right\rceil.$$

Proof. We have to show that

$$\text{ind}_{\mathbb{Z}_r}(B_0(\mathcal{S})) \leq (r - 1) \cdot \chi(\mathcal{S}) - 1.$$

A map $f_c : B_0(\mathcal{S}) \rightarrow \mathcal{C}_{\chi(\mathcal{S})}$ defined on the vertices of $B_0(\mathcal{S})$ is induced from a colouring $c : V(\mathcal{S}) \rightarrow [\chi(\mathcal{S})]$ via

$$(\emptyset, \dots, \emptyset, v, \emptyset, \dots, \emptyset) \mapsto (\emptyset, \dots, \emptyset, c(v), \emptyset, \dots, \emptyset),$$

where $v \in V(\mathcal{S})$ and a vertex of $B_0(\mathcal{S})$ that has non-empty entry in coordinate i is mapped to a vertex of $\mathcal{C}_{\chi(\mathcal{S})}$ that has non-empty entry in coordinate i . This map is well-defined since c is a colouring and maps simplices of $B_0(\mathcal{S})$ to simplices of $\mathcal{C}_{\chi(\mathcal{S})}$. In fact, $\text{Im } f_c$ is a subcomplex of $\mathcal{C}_{\chi(\mathcal{S})}$ that is invariant under the \mathbb{Z}_r -action of $\mathcal{C}_{\chi(\mathcal{S})}$ and f_c is \mathbb{Z}_r -equivariant, no matter whether \mathcal{S} is multiplicity-free or not. Hence, we have

$$\text{ind}_{\mathbb{Z}_r}(B_0(\mathcal{S})) \leq \text{ind}_{\mathbb{Z}_r}(\text{Im } f_c) \leq \dim(\text{Im } f_c) \leq (r - 1) \cdot \chi(\mathcal{S}) - 1$$

since a maximal simplex of $\text{Im } f_c$ contains at most $(r - 1) \cdot \chi(\mathcal{S})$ many vertices. \square

4.5 A COMBINATORIAL LOWER BOUND FOR KNESER HYPERGRAPHS WITH MULTIPLICITIES

We now give an alternative proof of Ziegler's result in case of constant s . We start with the non-degenerate case and prime r .

Theorem 4.5.1. If r is prime, s an integer with $1 \leq s < r$, and $\mathcal{T} \neq \emptyset$ we have

$$\chi(KG_s^r \mathcal{T}) \geq \left\lceil \frac{\text{ind}_{\mathbb{Z}_r}(B_0(KG_s^r \mathcal{T})) + 1}{r - 1} \right\rceil \geq \left\lceil \frac{\text{cd}_s^r \mathcal{T}}{r - 1} \right\rceil.$$

Proof. We show that $\text{cd}_s^r \mathcal{T} - 1 \leq \text{ind}_{\mathbb{Z}_r}(\mathbf{B}_0(\text{KG}_s^r \mathcal{T}))$ for all prime numbers r and apply Theorem 4.4.2. To do so, we define a map

$$\begin{aligned} g : 2^{[n]} &\longrightarrow 2^{\mathcal{T}} \\ U &\longmapsto \{T \in \mathcal{T} \mid T \subseteq U\}. \end{aligned}$$

This map is used to define another map

$$\begin{aligned} f : \mathbf{P}_s^r \mathcal{T} &\longrightarrow \text{sd}(\mathbf{B}_0(\text{KG}_s^r \mathcal{T}) \setminus \emptyset) \\ (U_1, \dots, U_r) &\longmapsto (g(U_1), \dots, g(U_r)). \end{aligned}$$

We now want to show that this map f is well-defined. If U_1, \dots, U_r are s -disjoint subsets of $[n]$ and $U'_i \subseteq U_i$ for $1 \leq i \leq r$, then U'_1, \dots, U'_r are certainly s -disjoint. Since at least one U_i contains an element of \mathcal{T} by definition of $\mathbf{P}_s^r \mathcal{T}$, we deduce that f is well-defined. Moreover, the map f is a simplicial map because chains of elements of $\mathbf{P}_s^r \mathcal{T}$ are mapped to chains of simplices of $\mathbf{B}_0(\text{KG}_s^r \mathcal{T})$. Finally, the map is \mathbb{Z}_r -equivariant and surjective, hence

$$\text{ind}_{\mathbb{Z}_r}(\mathbf{P}_s^r \mathcal{T}) \leq \text{ind}_{\mathbb{Z}_r}(\text{Im } f) = \text{ind}_{\mathbb{Z}_r}(\text{sd}(\mathbf{B}_0(\text{KG}_s^r \mathcal{T}) \setminus \emptyset)) = \text{ind}_{\mathbb{Z}_r}(\mathbf{B}_0(\text{KG}_s^r \mathcal{T})).$$

To apply Sarkaria's inequality, consider the subcomplex \mathbf{L} of \mathbf{P}_s^r that is induced by the vertices $V(\mathbf{P}_s^r) \setminus V(\mathbf{P}_s^r \mathcal{T})$. We have $\mathbf{P}_s^r \subseteq \mathbf{P}_s^r \mathcal{T} * \mathbf{L}$, since $\mathbf{P}_s^r \mathcal{T}$ and \mathbf{L} are subcomplexes of \mathbf{P}_s^r , but \mathbf{P}_s^r may not contain some simplices of $\mathbf{P}_s^r \mathcal{T} * \mathbf{L}$. Hence

$$\text{ind}_{\mathbb{Z}_r}(\mathbf{P}_s^r) \leq \text{ind}_{\mathbb{Z}_r}(\mathbf{P}_s^r \mathcal{T} * \mathbf{L}) \leq \text{ind}_{\mathbb{Z}_r}(\mathbf{P}_s^r \mathcal{T}) + \text{ind}_{\mathbb{Z}_r}(\mathbf{L}) + 1.$$

Since $\text{ind}_{\mathbb{Z}_r}(\mathbf{P}_s^r) = ns - 1$ and since the dimension is an upper bound for the index, we have

$$ns - 1 - \dim(\mathbf{L}) - 1 \leq \text{ind}_{\mathbb{Z}_r}(\mathbf{P}_s^r \mathcal{T}).$$

But $\dim(\mathbf{L}) \leq ns - \text{cd}_s^r \mathcal{T} - 1$: A simplex of dimension d in \mathbf{L} corresponds to a chain of length $d + 1$ in \mathbf{P}_s^r . But every chain of length larger than $ns - \text{cd}_s^r \mathcal{T}$ in \mathbf{P}_s^r contains at least one s -disjoint r -tuple (U_1, \dots, U_r) that satisfies $\sum_{i=1}^r |U_i| \geq ns - \text{cd}_s^r \mathcal{T} + 1$. Such a chain contains therefore an element of $\mathbf{P}_s^r \mathcal{T}$ and does not correspond to a simplex of \mathbf{L} . Altogether, we obtain

$$\text{cd}_s^r \mathcal{T} - 1 \leq \text{ind}_{\mathbb{Z}_r}(\mathbf{P}_s^r \mathcal{T}),$$

which proves the claim for the case that r is a prime. □

For the following induction we also have to analyse the degenerate cases.

Theorem 4.5.2. For integers r and s with $s \geq r > 1$ and $\mathcal{T} \neq \emptyset$, we have

$$\chi(\text{KG}_s^r \mathcal{T}) \geq \left\lceil \frac{\text{cd}_s^r \mathcal{T}}{r - 1} \right\rceil.$$

For integers r and s with $s \geq r > 1$ and $\mathcal{T} = \emptyset$, we have

$$\chi(\text{KG}_s^r \mathcal{T}) \geq \left\lceil \frac{\text{cd}_s^r \mathcal{T} - n(s - r)}{r - 1} \right\rceil.$$

For integers r and s with $r > s \geq 1$ and $\mathcal{T} = \emptyset$, we have

$$\chi(\text{KG}_s^r \mathcal{T}) \geq \left\lceil \frac{\text{cd}_s^r \mathcal{T}}{r-1} \right\rceil.$$

Proof. The degenerate cases are easily derived from the following observations:

- If $s \geq r$ and $\mathcal{T} \neq \emptyset$ then $\chi(\text{KG}_s^r \mathcal{T}) = \infty$ while $\text{cd}_s^r \mathcal{T} < \infty$.
- If $s \geq r$ and $\mathcal{T} = \emptyset$ then $\chi(\text{KG}_s^r \mathcal{T}) = 0$ and $\text{cd}_s^r \mathcal{T} = n(s-r)$.
- If $s < r$ and $\mathcal{T} = \emptyset$ then $\chi(\text{KG}_s^r \mathcal{T}) = \text{cd}_s^r \mathcal{T} = 0$.

□

We now discuss the induction used by Ziegler [69] to derive a statement for non-prime r from the prime case which we just proved. The idea of this induction traces back to Alon, Frankl, and Lovász [3] and was also applied by Kříž [41] and Matoušek [49] in case $s = 1$.

Before we discuss the details of the induction, we emphasise that the statements of Theorem 4.5.2 are true for non-prime r ($r \geq 2$).

Suppose now that r is not a prime, that is, $r = r'r''$ for some integers $r', r'' < r$, and that the claim has been shown for all positive integers less than r . Consider an s -disjoint r -uniform Kneser hypergraph $\text{KG}_s^r \mathcal{T}$ with multiplicities on the ground set $[n]$ and assume that

$$\text{cd}_s^r \mathcal{T} > (r-1)\chi(\text{KG}_s^r \mathcal{T}).$$

For a subset S of $[n]$ denote the elements of \mathcal{T} that are subsets of S by \mathcal{T}_S . We now consider an auxiliary set \mathcal{U} of subsets of $[n]$:

$$\mathcal{U} := \left\{ N \subseteq [n] \mid \text{cd}_1^{r'} \mathcal{T}_N > (r'-1)\chi(\text{KG}_s^r \mathcal{T}) \right\}.$$

For each $N \in \mathcal{U}$ we have by the result of Kříž [41] ($s = 1$ and arbitrary r'):

$$(r'-1)\chi(\text{KG}_1^{r'} \mathcal{T}_N) \geq \text{cd}_1^{r'} \mathcal{T}_N > (r'-1)\chi(\text{KG}_s^r \mathcal{T}).$$

Hence, we obtain

$$\chi(\text{KG}_1^{r'} \mathcal{T}_N) > \chi(\text{KG}_s^r \mathcal{T}) \quad \text{for each } N \in \mathcal{U}. \quad (4.1)$$

According to Ziegler [69], we now want to relate the chromatic number of $\text{KG}_s^{r''} \mathcal{U}$ to the chromatic number of $\text{KG}_s^r \mathcal{T}$. This is done as follows in case $\mathcal{U} \neq \emptyset$. We know from Ziegler [69, p. 680] or from Lemma 4.5.3 at the end of this section that

$$\text{cd}_s^{r''} \mathcal{U} > (r''-1)\chi(\text{KG}_s^r \mathcal{T}). \quad (4.2)$$

By induction we now have

$$(r''-1)\chi(\text{KG}_s^{r''} \mathcal{U}) \geq \text{cd}_s^{r''} \mathcal{U} > (r''-1)\chi(\text{KG}_s^r \mathcal{T}).$$

We note that these inequalities hold for $r'' < s$ and for $r'' \geq s$. Moreover, they imply

$$\chi(\text{KG}_s^{r''} \mathcal{U}) > \chi(\text{KG}_s^r \mathcal{T}). \quad (4.3)$$

Consider a colouring $c : \mathcal{T} \rightarrow [\chi(\text{KG}_s^r \mathcal{T})]$ of $\text{KG}_s^r \mathcal{T}$. For every set $N \in \mathcal{U}$ we find r' disjoint sets of $\mathcal{T}_{|N}$ that get the same colour $i \in [\chi(\text{KG}_s^r \mathcal{T})]$ by (4.1). Using this, we obtain a colouring $c' : \mathcal{U} \rightarrow [\chi(\text{KG}_s^r \mathcal{T})]$ which colours every node $N \in \mathcal{U}$ of $\text{KG}_s^{r''} \mathcal{U}$ in one of the colours i which c assigns to r' 1-disjoint sets of $\mathcal{T}_{|N}$. But (4.3) guarantees that there are r'' sets $N_j \in \mathcal{U}$ which are s -disjoint and coloured by c' with the same colour i_0 . Together, we obtain $r = r'r''$ s -disjoint sets in \mathcal{T} which are coloured by c with the colour i_0 . This contradicts the definition of $\chi(\text{KG}_s^r \mathcal{T})$ and c .

The proof of Lemma 4.5.3, that is, of (4.2), still works in case if $r'' > s$ and $\mathcal{U} = \emptyset$. By induction we conclude

$$(r'' - 1)\chi(\text{KG}_s^{r''} \mathcal{U}) \geq \text{cd}_s^{r''} \mathcal{U} > (r'' - 1)\chi(\text{KG}_s^r \mathcal{T}).$$

But this contradicts $\chi(\text{KG}_s^{r''} \mathcal{U}) = 0$, that is, $r'' > s$ and $\mathcal{U} = \emptyset$ does not occur.

An analysis of the case $r'' \leq s$ and $\mathcal{U} = \emptyset$ remains open.

We finish this section with a proof of (4.2).

Lemma 4.5.3. We use notation as above and assume that $\text{cd}_s^r \mathcal{T} > (r - 1)\chi(\text{KG}_s^r \mathcal{T})$. If $\mathcal{U} \neq \emptyset$ or $r'' > s$ we have

$$\text{cd}_s^{r''} \mathcal{U} > (r'' - 1)\chi(\text{KG}_s^r \mathcal{T}).$$

Proof. Suppose this inequality is not true. Then we find an s -disjoint family $N_1, \dots, N_{r''}$ of subsets of $[n]$ such that no N_j contains an element of \mathcal{U} and such that

$$\sum_{j=1}^{r''} |N_j| \geq n \cdot s - (r'' - 1)\chi(\text{KG}_s^r \mathcal{T}).$$

By definition of \mathcal{U} and the fact that no N_j is an element of \mathcal{U} , we have for all j

$$\text{cd}_1^{r'} \mathcal{T}_{|N_j} \leq (r' - 1)\chi(\text{KG}_s^r \mathcal{T}).$$

We therefore find for each j disjoint sets $M_{j1}, \dots, M_{jr'}$ such that no M_{jk} contains an element of \mathcal{T} and

$$\sum_{k=1}^{r'} |M_{jk}| \geq |N_j| - (r' - 1)\chi(\text{KG}_s^r \mathcal{T}).$$

Altogether we have $r - r'r''$ many sets $M_{jk} \subseteq [n]$ that form an s -disjoint family and none contains an element of \mathcal{T} . But

$$\begin{aligned} \sum_{j=1}^{r''} \sum_{k=1}^{r'} |M_{jk}| &\geq \sum_{j=1}^{r''} |N_j| - r''(r' - 1)\chi(\text{KG}_s^r \mathcal{T}) \\ &\geq n \cdot s - (r'' - 1)\chi(\text{KG}_s^r \mathcal{T}) - r''(r' - 1)\chi(\text{KG}_s^r \mathcal{T}) \\ &= n \cdot s - (r - 1)\chi(\text{KG}_s^r \mathcal{T}), \end{aligned}$$

which contradict the assumption that $\text{cd}_s^r \mathcal{T} > (r - 1)\chi(\text{KG}_s^r \mathcal{T})$. Hence we have shown that $\text{cd}_s^{r''} \mathcal{U} > (r'' - 1)\chi(\text{KG}_s^r \mathcal{T})$. \square

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